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Proceedings:

COMPUTATIONAL CHALLENGES IN THE RELIABILITY ASSESSMENT OF ENGINEERING STRUCTURES

January 24, 2018
Delft, The Netherlands

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PREFACE

Aging and deteriorating infrastructure is an urgent issue in all industrialized countries. As the built environment comprises a substantial part (~80%) of our national wealth it is crucial to address this issue. Many civil engineering structures are approaching the end of their intended design life, for example most of our transportation infrastructure has been built in the 1960s and 1970s. Assessing the reliability of these structures is essential to keep the existing stock in operation.

However, structural reliability and remaining service life assessment of these complex structures can be a daunting task. The main issue is that these assessments often involve a large number of random variables (e.g. due to random fields), have computationally expensive physical models (e.g. NL-FEM models) and have small failure probabilities (1e3 to 1e6). The reliability analysis of complex structures quickly becomes a computational challenge.

To face this challenge, The Department of Structural Reliability at TNO organized a workshop on this topic. The aim of the workshop was to bring together researchers, practitioners, and software developers from all over the world to share experience, learn from each other, and to jointly find ways of solving these challenges.

These proceedings contain the abstracts and slides of the 11 lectures held during the workshop. The first half of the lectures dealt with state-of-the-art reliability methods. The second half of the lectures dealt with the latest developments and challenges in engineering practice.

We believe that the workshop was a great success, with participants form 22 different affiliations and from 10 different countries; from the field of Civil Engineering and the field of Aerospace Engineering; from the academia and from the practice.
We would like to thank everyone who contributed to this workshop.

The organizing committee

LIST OF ATTENDEES

Chairman

Ton Vrouwenvelder	TNO & Delft University of Technology

Organizing Committee

Árpád Rózsás	TNO
Arthur Slobbe	TNO
Nadieh Meinen	TNO
Agnieszka Bigaj van Vliet	TNO
Raphaël Steenbergen	TNO & Ghent University

Speakers

Iason Papaioannou	Technical University of Munich
Bruno Sudret	ETH Zurich
Edoardo Patelli	University of Liverpool
Ziqi Wang	Guangzhou University
Karl Breitung	Technical University of Munich
Timo Schweckendiek	Deltares & Delft University of Technology
Frank Grooteman	National Aerospace Laboratory NLR
Max Hendriks	Delft University of Technology & Norwegian University of Science and Technology
Bram van den Eijnden	Delft University of Technology
Vladimir Červenka	Červenka Consulting

Registrants

Antony van Middelkoop	ABT b.v.
Job Janssen	ABT b.v.
Coen van der Vliet	Arcadis
Niels Kostense	Arcadis
Rudy Chocat	ArianeGroup / Cenaero / UTC
Rein de Vries	Arup
Albrecht Schmidt	Bauhaus-Universität Weimar, Institute of Structural Mechanics
Viktor Budaházy	BME Department of Structural Engineering
Jan Mašek	Brno University of Technology
Miroslav Vořechovský	Brno University of Technology
Mark van der Krogt	Delft University of Technology
Pieter Dolron	Delft University of Technology
Quanxin Jiang	Delft University of Technology
Tianxiang Wang	Delft University of Technology
Hannah Suh Heo	Delft University of Technology & Sweco GmbH
Ana Teixeira	Deltares
Jonathan Nuttall	Deltares
Rob Brinkman	Deltares
Panos Evangeliou	DIANA FEA B.V.
Gerd-Jan Schreppers	DIANA FEA BV
Marco Broccardo	ETH Zurich
Didier Droogné	Ghent University
Robby Caspeele	Ghent University
Ruben Van Coile	Ghent University
Wouter Botte	Ghent University
Vincent Chabridon	ONERA - The French Aerospace Lab
Anita Laera	Plaxis b.v.
Elena Lazovik	TNO
Erik Langius	TNO
Henco G. Burggraaf	TNO
Jaap Weerheijm	TNO
Liesette la Gasse	TNO
Wim Courage	TNO
Alan O'Connor	Trinity College Dublin
Tobias-Emanuel Regenhardt	University Hannover
Anaïs Couasnon	Vrije Universiteit Amsterdam
Anton van der Meer	Witteveen + Bos
Richard Roggeveld	Witteveen + Bos

CONTENTS

PART 1: RELIABILITY METHODS

PART 2: PRACTICE

PART 1: RELIABILITY METHODS

SEQUENTIAL SAMPLING APPROACHES FOR RELIABILITY ASSESSMENT

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Structural reliability analysis requires estimation of the probability of failure, which is defined through a potentially high dimensional probability integral. The failure event is expressed in terms of an (often complex) engineering model with uncertain input. The probability of failure is commonly estimated with Monte Carlo-based sampling approaches due to their robustness in dealing with complex numerical models. Although the performance of the Monte Carlo method does not depend on the dimension of the random variable space, it deteriorates geometrically with decrease of the target failure probability.

In this talk, a number of advanced sampling methods are discussed that improve the efficiency of crude Monte Carlo, while maintaining to a certain extent its independency on the number of random variables. In particular, we discuss methods that perform a sequence of sampling steps with aim at obtaining samples from a theoretically optimal importance sampling density – the density of the random variables censored at the failure domain. These methods include subset simulation [1, 2], sequential importance sampling [3] and cross-entropy importance sampling [4,5]. We focus on the former two and discuss computational settings that optimize their performance in high dimensional problems. We additionally discuss the potential of using surrogate or multi-fidelity models within a sequential approach to enhance computational efficiency. The performance of the methods is demonstrated with a number of numerical examples in high dimensions.

References:

[1] Au, S. K., & Beck, J. L. (2001). Estimation of small failure probabilities in high dimensions by subset simulation. Probabilistic Engineering Mechanics, 16(4), 263-277.

[2] Papaioannou, I., Betz, W., Zwirglmaier, K., & Straub, D. (2015). MCMC algorithms for subset simulation. Probabilistic Engineering Mechanics, 41, 89-103.

[3] Papaioannou, I., Papadimitriou, C., & Straub, D. (2016). Sequential importance sampling for structural reliability analysis. Structural safety, 62, 66-75.

[4] Wang, Z., & Song, J. (2016). Cross-entropy-based adaptive importance sampling using von Mises-Fisher mixture for high dimensional reliability analysis. Structural Safety, 59, 42-52.

[5] Papaioannou, I., Geyer, S., & Straub, D. Modified cross-entropy-based importance sampling with a flexible mixture model. Manuscript.

Sequential sampling approaches for reliability assessment

TU Delft, 24 January 2018

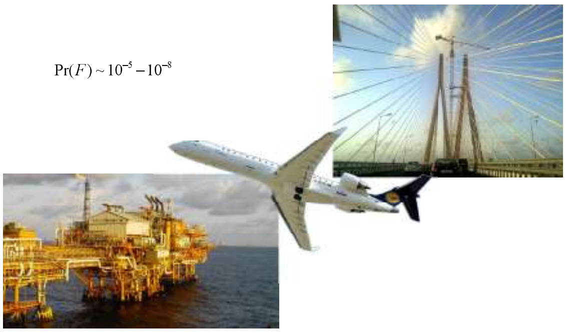
Iason Papaioannou

Engineering Risk Analysis Group, TU München

Reliability analysis

Estimation of rare event probabilities

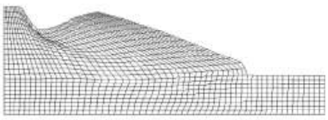
$\Pr(F) \sim 10^{-5} - 10^{-8}$

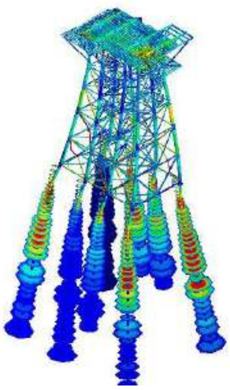


Sources: Daniel Straub, Satish Krishnamurthy, ASDFGH

Estimating the probability of failure

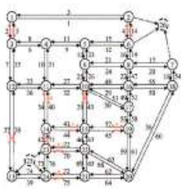
- Models of engineering systems
- Parameters modeled as random variables
- Enables extrapolation to extreme situations

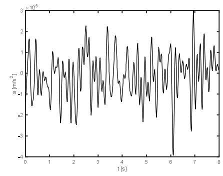


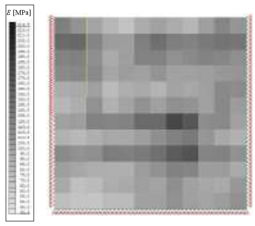


High dimensional inputs

- Systems with large numbers of component
- Time/space variable inputs





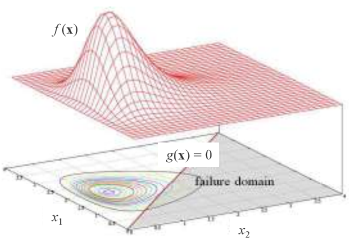


Reliability analysis

- Random variables $\mathbf{X} = [X_1, X_2, \ldots, X_n]^{\mathrm{T}}$

  Joint PDF: $f(\mathbf{x})$

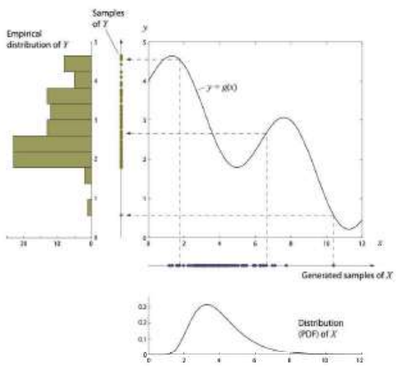
- Failure condition defined through limit-state function $g(\mathbf{x})$ s.t. $F = \{g(\mathbf{x}) \leq 0\}$



- Probability of failure: $P_F := \Pr(F) = \int_{g(\mathbf{x}) \leq 0} f(\mathbf{x}) d\mathbf{x}$

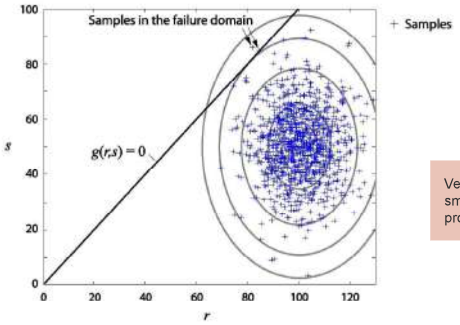
Simulation methods

Based on Monte Carlo simulation



- Robust: Can deal with complex numerical models
- Efficiency does not depend on the dimension of the problem

Monte Carlo for reliability analysis



Very inefficient for small failure probabilities!!

Monte Carlo

Probability of failure

$$P_F = \int_{g(\mathbf{x})\le 0} f(\mathbf{x})\,d\mathbf{x} = \int_{\mathrm{R}^n} I\big(g(\mathbf{x})\le 0\big) f(\mathbf{x})\,d\mathbf{x} = \mathrm{E}_f\big[I\big(g(\mathbf{x})\le 0\big)\big]$$

Indicator function

$$I\big(g(\mathbf{x})\le 0\big) = \begin{cases} 1 & \text{if } g(\mathbf{x})\le 0 \\ 0 & \text{otherwise} \end{cases}$$

Estimate of probability

$$\hat{P}_F = \hat{\mathrm{E}}_f\big[I\big(g(\mathbf{x})\le 0\big)\big] = \frac{1}{n_s}\sum_{k=1}^{n_s} I\big(g(\mathbf{x}_k)\le 0\big)$$

Coefficient of variation of estimate

$$CV_{\hat{P}_F} = \frac{\sqrt{\mathrm{Var}\big(\hat{P}_F\big)}}{\mathrm{E}\big[\hat{P}_F\big]} = \sqrt{\frac{1-P_F}{n_s P_F}}$$

Importance sampling

Probability of failure

$$P_F = \int_{g(\mathbf{x})\le 0} \frac{f(\mathbf{x})}{h(\mathbf{x})} h(\mathbf{x})\,d\mathbf{x} = \int_{\mathrm{R}^n} I\big(g(\mathbf{x})\le 0\big) w(\mathbf{x}) h(\mathbf{x})\,d\mathbf{x} = \mathrm{E}_h\big[I\big(g(\mathbf{x})\le 0\big) w(\mathbf{x})\big]$$

Importance sampling function: $h(\mathbf{x})$

Importance weight function: $w(\mathbf{x}) = \dfrac{f(\mathbf{x})}{h(\mathbf{x})}$

Estimate of probability

$$\hat{P}_F = \hat{\mathrm{E}}_h\big[I\big(g(\mathbf{x})\le 0\big) w(\mathbf{x})\big] = \frac{1}{n_s}\sum_{k=1}^{n_s} I\big(g(\mathbf{x}_k)\le 0\big) w(\mathbf{x}_k)$$

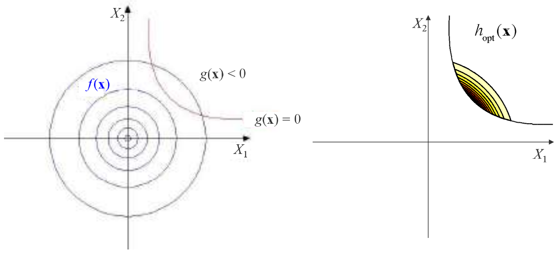
Variance of estimate

$$\mathrm{Var}\big(\hat{P}_F\big) = \frac{1}{n_s}\Big(\mathrm{E}_h\big[I\big(g(\mathbf{x})\le 0\big) w(\mathbf{x})^2\big] - P_F^2\Big)$$

Importance sampling (II)

Optimal importance sampling density

$$h_{\mathrm{opt}}(\mathbf{x}) = \frac{1}{P_F} I\big(g(\mathbf{x})\le 0\big) f(\mathbf{x})$$



Requires the knowledge of P_F

Importance sampling (III)

Typical choice of IS density

- Gaussian density centered at FORM design point $\varphi(\mathbf{x}-\mathbf{x}_0)$

  Importance weight function: $w(\mathbf{x}) = \dfrac{f(\mathbf{x})}{\varphi(\mathbf{x}-\mathbf{x}_0)} \to 0$ for $n \to \infty$

Reduced efficiency in high dimensions
[Au & Beck 2003, Katafygiotis & Zuev 2007]

Advanced sampling methods

- Line sampling [Hohenbichler & Rackwitz 1988; Koutsourelakis et al. 2004]
- Subset simulation [Au & Beck 2001]
- Asymptotic sampling [Bucher 2009]
- Sequential importance sampling
  [Beaurepaire et al. 2013; Papaioannou et al. 2016]
- Cross-entropy based importance sampling
  [Rubinstein 2001; Kurtz & Song 2013; Wang & Song 2016]

Sequential sampling approaches

- Sample a sequence of distributions that gradually approximate the desired distribution
- Sequential sampling for Bayesian analysis/statistical physics
  - Annealed importance sampling [Neal 2001]
  - Particle filter/Resample-move algorithms [Chopin 2002]
  - Sequential Monte Carlo [Del Moral et al. 2004, 2006]
  - Transitional MCMC [Ching & Chen 2007; Betz et al. 2016]
- **Sequential sampling for reliability analysis**
  - Subset simulation [Au & Beck 2001]
  - Sequential importance sampling [Beaurepaire et al. 2013; Papaioannou et al. 2016]
  - Cross-entropy method [Rubinstein 2001; Kurtz & Song 2013; Wang & Song 2016]

Sequential sampling approaches for reliability analysis

Sequential importance sampling

- Consider a sequence of distributions $\{h_j(\mathbf{x}), j = 1,\ldots,m\}$ such that

$$h_j(\mathbf{x}) = f(\mathbf{x}) \quad \text{and} \quad h_j(\mathbf{x}) = h_{\text{opt}}(\mathbf{x})$$

- Each distribution is known up to a normalizing constant

$$h_j(\mathbf{x}) = \frac{\eta_j(\mathbf{x})}{P_j}$$

- We want to sample each distribution $h_j(\mathbf{x})$ and estimate the normalizing constants P_j

Sequential importance sampling (II)

Estimate P_j with importance sampling and IS density $h_{j-1}(\mathbf{x})$

$$P_j = \int_{\mathrm{R}^n} \eta_j(\mathbf{x}) d\mathbf{x} = P_{j-1} \int_{\mathrm{R}^n} \frac{\eta_j(\mathbf{x})}{\eta_{j-1}(\mathbf{x})} h_{j-1}(\mathbf{x}) d\mathbf{x}$$

$$\rightarrow \quad \frac{P_j}{P_{j-1}} = \int_{\mathrm{R}^n} \frac{\eta_j(\mathbf{x})}{\eta_{j-1}(\mathbf{x})} h_{j-1}(\mathbf{x}) d\mathbf{x} = \mathrm{E}_{h_{j-1}}\left[w_j(\mathbf{x})\right]$$

where $w_j(\mathbf{x}_k) = \frac{\eta_j(\mathbf{x}_k)}{\eta_{j-1}(\mathbf{x}_k)}$

Estimate of ratio of normalizing constants

$$\hat{S}_j = \frac{\hat{P}_j}{\hat{P}_{j-1}} = \frac{1}{n_s}\sum_{k=1}^{n_s} w_j(\mathbf{x}_k)$$

where $\mathbf{x}_k \sim h_{j-1}(\mathbf{x})$

Sequential importance sampling (III)

Sample each distribution $h_j(\mathbf{x})$

- Obtain weighted samples form $h_j(\mathbf{x})$ using samples from $h_{j-1}(\mathbf{x})$

If $\mathbf{x}_k \sim h_{j-1}(\mathbf{x})$ then $(\mathbf{x}_k, w_j(\mathbf{x}_k)) \sim h_j(\mathbf{x})$

where $w_j(\mathbf{x}_k) = \frac{\eta_j(\mathbf{x}_k)}{\eta_{j-1}(\mathbf{x}_k)}$

- Resample $(\mathbf{x}_k, w_j(\mathbf{x}_k))$ to obtain uniformly weighted samples of $h_j(\mathbf{x})$
- Move the samples applying MCMC with invariant distribution $h_j(\mathbf{x})$

Distribution sequences for reliability analysis

Optimal IS density

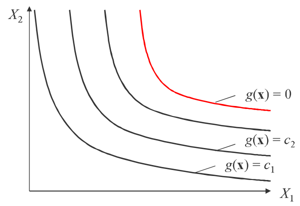
$$h_{\text{opt}}(\mathbf{x}) = \frac{1}{P_F} I\big(g(\mathbf{x}) \le 0\big) f(\mathbf{x})$$

Subset simulation [Au & Beck 2001]

Define a sequence of densities:

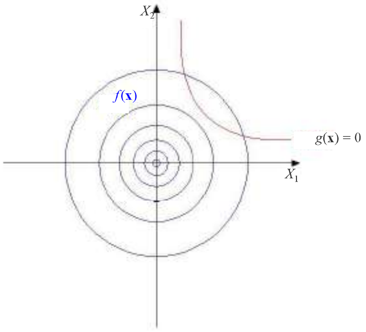
$$h_j(\mathbf{x}) = \frac{1}{P(F_j)} I_{F_j}(\mathbf{x}) f(\mathbf{x}) \quad \text{where} \quad F_0 \supset F_1 \supset \cdots \supset F_M = F$$

Intermediate failure domain: $F_j = \left\{g(\mathbf{x}) \le c_j\right\}$ with $\infty = c_0 > c_1 > \cdots c_M = 0$



Illustration

$g(\mathbf{x}) = 0.1(x_1 - x_2)^2 - \frac{1}{\sqrt{2}}(x_1 - x_2) + 2.5 \qquad \mathbf{X} \sim N(\mathbf{0}, \mathbf{I})$

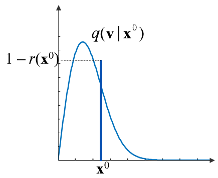


Metropolis-Hastings algorithm

Metropolis-Hastings algorithm for sampling from $h_j(\mathbf{x}) \propto I_{F_j}(\mathbf{x}) f(\mathbf{x})$

M-H transition density

$$p(\mathbf{v} \mid \mathbf{x}) = \alpha(\mathbf{x}, \mathbf{v}) q(\mathbf{v} \mid \mathbf{x}) + (1 - r(\mathbf{x})) \delta_{\mathbf{x}}(\mathbf{v})$$



Proposal density: $q(\mathbf{v} \mid \mathbf{x})$

Acceptance probability of candidate:

$$\alpha(\mathbf{x}, \mathbf{v}) = I_{F_j}(\mathbf{y}) \min\left\{1, \frac{f(\mathbf{v}) q(\mathbf{x} \mid \mathbf{v})}{f(\mathbf{x}) q(\mathbf{v} \mid \mathbf{x})}\right\}$$

Probability that the chain moves from the current state:

$$r(\mathbf{x}) = \int_{\mathbf{v} \in \mathbb{R}^n} \alpha(\mathbf{x}, \mathbf{v}) q(\mathbf{v} \mid \mathbf{x}) d\mathbf{v}$$

Dirac mass at $\mathbf{x}$: $\delta_{\mathbf{x}}(\mathbf{v})$

Random walk sampler

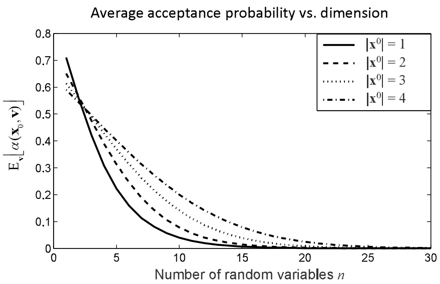
Proposal density chosen as Gaussian density centered at current state:

$$q(\mathbf{v} \mid \mathbf{x}) = \varphi(\mathbf{v} - \mathbf{x})$$

Acceptance probability for independent $f(\mathbf{x})$

$$\alpha(\mathbf{x}, \mathbf{v}) = I_{F_j}(\mathbf{v}) \min\left\{1, \frac{f(\mathbf{v})}{f(\mathbf{x})}\right\}$$

Example: Sampling from a Gaussian target



Low acceptance rate (reduced efficiency) in high dimensions
[Au & Beck 2001, Katafygiotis & Zuev 2007, Papaioannou et al. 2015]

Efficient samplers for high dimensions

- Component-wise (single component) Metropolis algorithm [Au & Beck 2001]
- **Conditional sampling (CS) algorithm** [Papaioannou et al. 2015, Au & Patelli 2016]

Conditional sampling (CS) algorithm

Choose $q(. \mid \mathbf{x}_0)$ as the multivariate Gaussian conditional on the current state $\mathbf{x}_0$:

$$q(\mathbf{v} \mid \mathbf{x}_0) = \varphi\left(\mathbf{v} - \rho \mathbf{x}_0, (1 - \rho^2) \mathbf{I}\right)$$

where ρ: correlation coefficient of the current with the candidate state

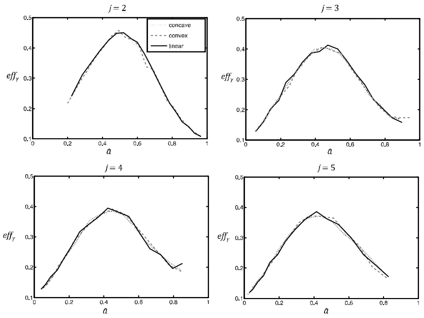
If $f(\mathbf{x})$ is Gaussian $\Longrightarrow$ $\alpha(\mathbf{x}_0, \mathbf{v}) = I_{F_j}(\mathbf{v})$

Efficiency is independent of the random dimension!

Papaioannou I., Betz W., Zwirglmaier K., Straub D.: MCMC algorithms for subset simulation. *Probabilistic Engineering Mechanics*, 41: 89-103

Adaptive CS algorithm [Papaioannou et al. 2015]

Choose ρ adaptively to match a near-optimal acceptance probability $\alpha^* = 0.44$



Papaioannou I., Betz W., Zwirglmaier K., Straub D.: MCMC algorithms for subset simulation. *Probabilistic Engineering Mechanics*, 41: 89-103

SuS: Effect of the MCMC sampler

1-D diffusion problem:

$$\frac{d}{dx}\left(a(z)\frac{dv}{dx}\right)=1,\ z\in[0,1] \quad \text{with} \quad v(0)=0,\ \left.\frac{dv}{dz}\right|_{x=1}=0$$

Log-diffusivity: Gaussian RF

Autocorrelation function:

$$\rho(z,z')=\exp\left(-|z-z'|/r\right); \quad r=0.01$$

Karhunen-Loève expansion with 200 terms:

$$\log a(z)=\mu_{\log a}+\sum_{i=1}^{200}\sqrt{\lambda_i}\varphi_i(z)x_i \qquad \mathbf{X}\sim N(\mathbf{0},\mathbf{I})$$

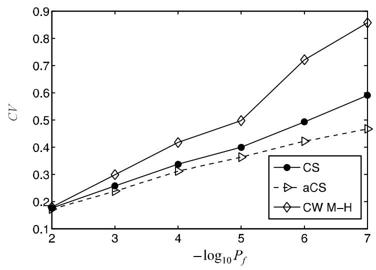
Spatial domain discretized by 100 piecewise-linear FEs

Limit state function: $g(\mathbf{x})=v_{\max}-v(\mathbf{x},z=1)$

SuS: Effect of the MCMC sampler

1-D diffusion problem:

$$\frac{d}{dx}\left(a(z)\frac{dv}{dx}\right)=1,\ z\in[0,1] \quad \text{with} \quad v(0)=0,\ \left.\frac{dv}{dz}\right|_{x=1}=0$$



Papaioannou I., Betz W., Zwirglmaier K., Straub D. (2015). MCMC algorithms for subset simulation. *Probabilistic Engineering Mechanics*, 41: 89-103

Distribution sequences for reliability analysis (II)

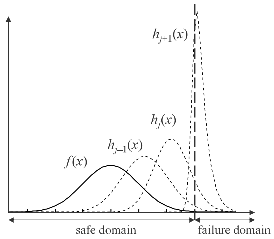
Optimal IS density

$$h_{\text{opt}}(\mathbf{x})=\frac{1}{P_F}I\left(g(\mathbf{x})\le 0\right)f(\mathbf{x})$$

Sequential importance sampling [Beaurepaire et al. 2013; Papaioannou et al. 2016]

Define a sequence of densities:

$$h_j(\mathbf{x})=\frac{1}{P_j}\Phi\left(-\frac{g(\mathbf{x})}{\sigma_j}\right)f(\mathbf{x}) \qquad \text{where} \qquad \infty=\sigma_1>\cdots>\sigma_M>0$$



MCMC sampling for SIS

- Conditional sampling algorithm for high dimensional problems
- Independent Metropolis-Hastings in low to moderate dimensional component and system reliability problems

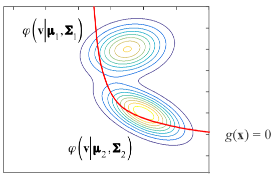
Papaioannou I., Papadimitriou C., Straub D. (2016). Sequential importance sampling for structural reliability analysis. *Structural Safety*, 62: 66-75.

Independent Metropolis-Hastings with Gaussian mixture proposal

Gaussian mixture proposal:

$$\pi(\mathbf{v})=\sum_{i=1}^{K}p_i\varphi\left(\mathbf{v}\,|\,\boldsymbol{\mu}_i,\boldsymbol{\Sigma}_i\right)$$

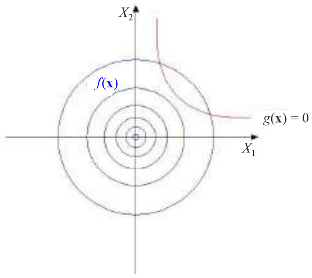
where $p_i,\boldsymbol{\mu}_i,\boldsymbol{\Sigma}_i$: are estimated using the weighted samples through application of the Expectation-Maximization algorithm



Papaioannou I., Papadimitriou C., Straub D. (2016). Sequential importance sampling for structural reliability analysis. *Structural Safety*, 62: 66-75.

Illustration

$$g(\mathbf{x}) = 0.1\left(x_1 - x_2\right)^2 - \frac{1}{\sqrt{2}}\left(x_1 - x_2\right) + 2.5 \qquad \mathbf{X} \sim N(\mathbf{0}, \mathbf{I})$$



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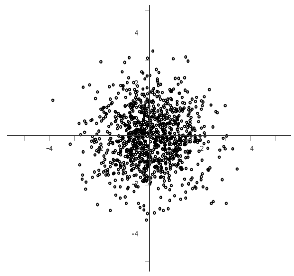
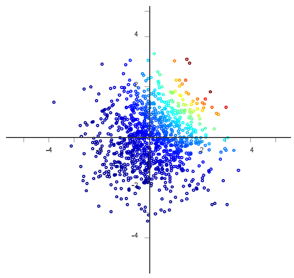


Illustration

$$g(\mathbf{x}) = 0.1\left(x_1 - x_2\right)^2 - \frac{1}{\sqrt{2}}\left(x_1 - x_2\right) + 2.5 \qquad \mathbf{X} \sim N(\mathbf{0}, \mathbf{I})$$



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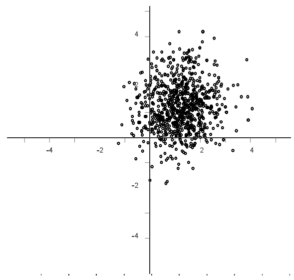

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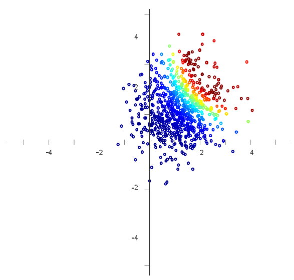

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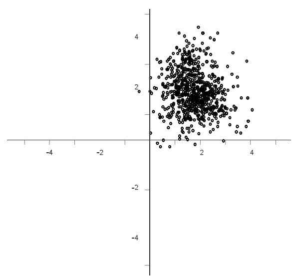

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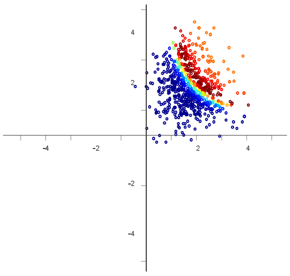

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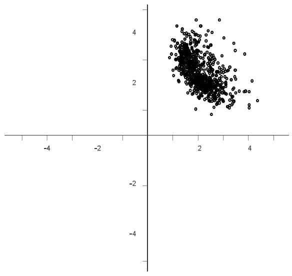
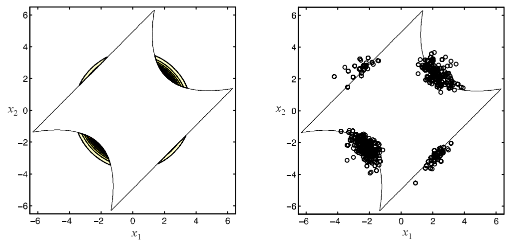


Performance in multi-modal failure domains

Series system reliability problem [Waarts 2000]

$$g(\mathbf{x}) = \min\left\{\begin{array}{c} 0.1(x_1 - x_2)^2 - (x_1 + x_2)/\sqrt{2} + 3 \\ 0.1(x_1 - x_2)^2 + (x_1 + x_2)/\sqrt{2} + 3 \\ x_1 - x_2 + 7/\sqrt{2} \\ x_2 - x_1 + 7/\sqrt{2} \end{array}\right\} \qquad \mathbf{X} \sim N(\mathbf{0}, \mathbf{I})$$



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Reference value $P_f = 2.2 \times 10^{-3}$

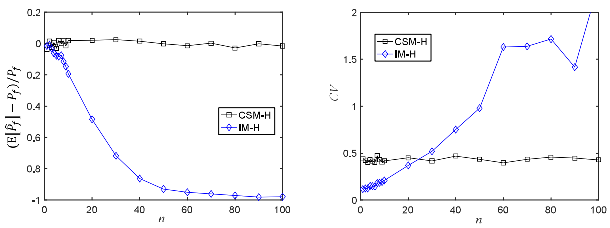
Number of samples per level n_s	SuS		SIS ($K = 4$)		SIS ($K = 10$)	
	Mean estimate	CV	Mean estimate	CV	Mean estimate	CV
500	2.27×10^{-3}	33%	1.84×10^{-3}	21%	1.57×10^{-3}	30%
1000	2.21×10^{-3}	22%	1.99×10^{-3}	13%	1.83×10^{-3}	16%
2000	2.23×10^{-3}	15%	2.10×10^{-3}	11%	2.01×10^{-3}	11%

Performance in high dimensions

Linear limit-state function in high dimensions [Engelund & Rackwitz 1993]

$$g(\mathbf{x}) = \beta\sqrt{n} - \sum_{i=1}^{n} x_i \qquad \mathbf{X} \sim N(\mathbf{0}, \mathbf{I})$$

Reference value $\beta = 3.5$; $P_f = 2.33 \times 10^{-4}$



Performance in high dimensions

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Reference value $\beta = 3.5$; $P_f = 2.33 \times 10^{-4}$

Number of random variables n	SuS		SIS (CSM-H)	
	Mean estimate	CV	Mean estimate	CV
10	2.34×10^{-4}	29%	2.32×10^{-4}	41%
100	2.34×10^{-4}	28%	2.29×10^{-4}	42%
1000	2.33×10^{-4}	28%	2.27×10^{-4}	42%

Observations

Subset simulation (SuS)

- Allows using only a fraction of samples from each previous distribution in the sequence
- MCMC within SuS does not require burn-in
- Efficient MCMC algorithms allow application to very high-dimensional problems

SIS with smooth transitions

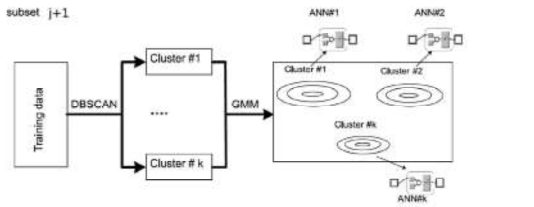
- Allows using all (weighted) samples from each previous distribution in the sequence to fit optimal MCMC proposals
- Has optimal performance in low- to moderate-dimensional problems

Problems of SIS/SuS

- No reliable estimate of the accuracy of the probability estimate exists
- The probability estimate becomes skewed with decrease of the target failure probability

Approaches for reducing computational cost

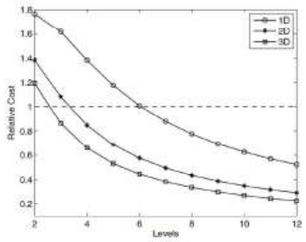
- Adaptive surrogate model representations, e.g. polynomial chaos expansions, artificial neural networks,…



Giovanis D. G., Papaioannou I., Straub D., Papadopoulos V. (2017). Bayesian updating with subset simulation using artificial neural networks. *Comput. Methods Appl. Mech. Eng.*, 319: 124-145

Approaches for reducing computational cost

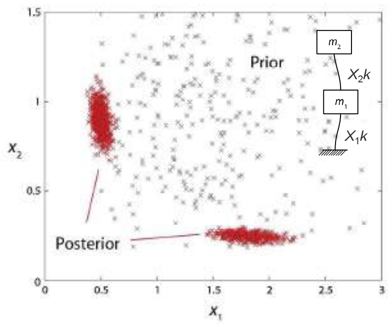
- Multi-level/multi-fidelity methods



Ullmann E., Papaioannou I. (2015). Multilevel estimation of rare events. *SIAM/ASA Journal of Uncertainty Quantification*, 3: 922-953

Bayesian analysis

Application of sampling-based approaches



Straub D., Papaioannou I. (2015). Bayesian updating with structural reliability methods. *Journal of Engineering Mechanics, ASCE*, 141(3): 04014134.

Summary

- Sequential sampling approaches for reliability analysis in high dimensions
- Based on sampling from a sequence of distribution that gradually approach a target sampling density
- SIS with smooth transitions performs well in low to medium dimensional problems
- SuS remains the optimal choice for high dimensional problems

Sequential sampling approaches for reliability assessment

TU Delft, 24 January 2018

Iason Papaioannou

Engineering Risk Analysis Group, TU München

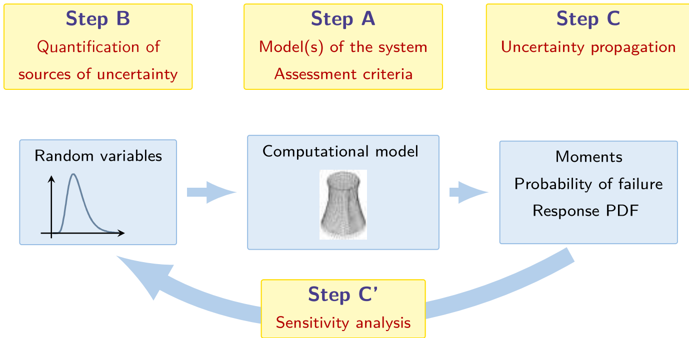
ACTIVE LEARNING METHODS FOR RELIABILITY ANALYSIS OF ENGINEERING SYSTEMS

Bruno Sudret
sudret@ibk.baug
Department of Civil, Environmental and Geomatic engineering

Active learning methods for reliability analysis of engineering systems

B. Sudret

Global framework for uncertainty quantification



B. Sudret, *Uncertainty propagation and sensitivity analysis in mechanical models – contributions to structural reliability and stochastic spectral methods (2007)*

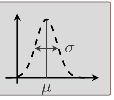
Step C: uncertainty propagation

Goal: estimate the uncertainty / variability of the quantities of interest (QoI) $Y = \mathcal{M}(\boldsymbol{X})$ due to the input uncertainty $f_{\boldsymbol{X}}$

- Output statistics, *i.e.* mean, standard deviation, etc.

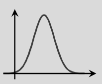
$$\mu_Y = \mathbb{E}_{\boldsymbol{X}}\left[\mathcal{M}(\boldsymbol{X})\right]$$
$$\sigma_Y^2 = \mathbb{E}_{\boldsymbol{X}}\left[\left(\mathcal{M}(\boldsymbol{X}) - \mu_Y\right)^2\right]$$

Mean/std. deviation



- Distribution of the QoI

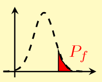
Response PDF



- Probability of exceeding an admissible threshold y_{adm}

$$P_f = \mathbb{P}\left(Y \geq y_{adm}\right)$$

Probability of failure



Limit state function

- For the assessment of the system's performance, failure criteria are defined, *e.g.* :

$$\text{Failure} \quad \Leftrightarrow \quad QoI = \mathcal{M}(\boldsymbol{x}) \geq q_{adm}$$

Examples:

+ admissible stress / displacements in civil engineering
+ max. temperature in heat transfer problems
+ crack propagation criterion in fracture mechanics

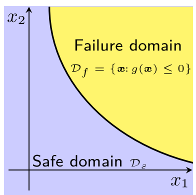
- The failure criterion is cast as a limit state function (performance function) $g: \boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}} \mapsto \mathbb{R}$ such that:

$g\left(\boldsymbol{x}, \mathcal{M}(\boldsymbol{x})\right) \leq 0$ Failure domain $\mathcal{D}_f$

$g\left(\boldsymbol{x}, \mathcal{M}(\boldsymbol{x})\right) > 0$ Safety domain $\mathcal{D}_s$

$g\left(\boldsymbol{x}, \mathcal{M}(\boldsymbol{x})\right) = 0$ Limit state surface

e.g. $g(\boldsymbol{x}) = q_{adm} - \mathcal{M}(\boldsymbol{x})$



Probability of failure

Definition

$$P_f = \mathbb{P}\left(\{\boldsymbol{X} \in D_f\}\right) = \mathbb{P}\left(g\left(\boldsymbol{X}, \mathcal{M}(\boldsymbol{X})\right) \leq 0\right)$$

$$P_f = \int_{\mathcal{D}_f = \{\boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}}\,:\, g(\boldsymbol{x}, \mathcal{M}(\boldsymbol{x})) \leq 0\}} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x}$$

Features

- Multidimensional integral, whose dimension is equal to the number of basic input variables $M = \dim \boldsymbol{X}$
- Implicit domain of integration defined by a condition related to the sign of the limit state function:

$$\mathcal{D}_f = \{\boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}} : \quad g(\boldsymbol{x}, \mathcal{M}(\boldsymbol{x})) \leq 0\}$$

- Failures are (usually) rare events: sought probability in the range 10^{-2} to 10^{-8}

Outline

Surrogate models for uncertainty quantification

A surrogate model $\tilde{\mathcal{M}}$ is an approximation of the original computational model $\mathcal{M}$ with the following features:

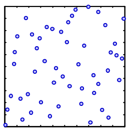
- It is built from a limited set of runs of the original model $\mathcal{M}$ called the experimental design $\mathcal{X} = \left\{ \boldsymbol{x}^{(i)},\, i = 1, \dots, n \right\}$
- It assumes some regularity of the model $\mathcal{M}$ and some general functional shape

Name	Shape	Parameters
Polynomial chaos expansions	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} a_{\boldsymbol{\alpha}}\, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{x})$	$\boldsymbol{a}_{\boldsymbol{\alpha}}$
Low-rank tensor approximations	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{l=1}^{R} b_l \left(\prod_{i=1}^{M} v_l^{(i)}(x_i) \right)$	$b_l,\, z_{k,l}^{(i)}$
Kriging (a.k.a Gaussian processes)	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \boldsymbol{\beta}^{\mathsf{T}} \cdot \boldsymbol{f}(\boldsymbol{x}) + Z(\boldsymbol{x}, \omega)$	$\boldsymbol{\beta},\, \sigma_Z^2,\, \boldsymbol{\theta}$
Support vector machines	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{i=1}^{n} a_i\, K(\boldsymbol{x}_i, \boldsymbol{x}) + b$	$\boldsymbol{a},\, b$

- It is fast to evaluate

Ingredients for building a surrogate model

- Select an experimental design $\mathcal{X}$ that covers at best the domain of input parameters: Latin hypercube sampling (LHS), low-discrepancy sequences



- Run the computational model $\mathcal{M}$ onto $\mathcal{X}$ exactly as in Monte Carlo simulation
- Smartly post-process the data $\{\mathcal{X},\, \mathcal{M}(\mathcal{X})\}$ through a learning algorithm

Name	Learning method
Polynomial chaos expansions	sparse grid integration, least-squares, compressive sensing
Low-rank tensor approximations	alternate least squares
Kriging	maximum likelihood, Bayesian inference
Support vector machines	quadratic programming

Gaussian process modelling

Gaussian process modelling (*a.k.a.* Kriging) assumes that the map $y = \mathcal{M}(\boldsymbol{x})$ is a realization of a Gaussian process:

$$Y(\boldsymbol{x}, \omega) = \sum_{j=1}^{p} \beta_j\, f_j(\boldsymbol{x}) + \sigma\, Z(\boldsymbol{x}, \omega)$$

where:

- $\boldsymbol{f} = \{f_j, j = 1, \dots, p\}^{\mathsf{T}}$ are predefined (*e.g.* polynomial) functions which form the trend or regression part
- $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_p\}^{\mathsf{T}}$ are the regression coefficients
- σ^2 is the variance of $Y(\boldsymbol{x}, \omega)$
- $Z(\boldsymbol{x}, \omega)$ is a stationary, zero-mean, unit-variance Gaussian process

$$\mathbb{E}\left[Z(\boldsymbol{x}, \omega)\right] = 0 \qquad \mathrm{Var}\left[Z(\boldsymbol{x}, \omega)\right] = 1 \qquad \forall\, \boldsymbol{x} \in \mathbb{X}$$

The Gaussian measure artificially introduced is different from the aleatory uncertainty on the model parameters $\boldsymbol{X}$

Assumptions on the trend and the zero-mean process

Prior assumptions are made based on the existing knowledge on the model to surrogate (linearity, smoothness, etc.)

Trend

- Simple Kriging: known constant β
- Ordinary Kriging: $p = 1$, unknown constant β
- Universal Kriging: f_j's is a set of *e.g.* polynomial functions, *e.g.* $\left\{ f_j(x) = x^{j-1},\, j = 1, \dots, p \right\}$ in 1D

Type of auto-correlation function of $Z(\boldsymbol{x})$

A family of auto-correlation function $R(\cdot;\, \boldsymbol{\theta})$ is selected:

$$\mathrm{Cov}\left[Z(\boldsymbol{x}), Z(\boldsymbol{x}')\right] = \sigma^2\, R(\boldsymbol{x}, \boldsymbol{x}';\, \boldsymbol{\theta})$$

e.g. square exponential, generalized exponential, Matérn, etc.

Matérn autocorrelation function (1D)

Definition

$$R_1(x, x') = \frac{1}{2^{\nu-1}\Gamma(\nu)} \left(\sqrt{2\,\nu}\, \frac{|x - x'|}{\ell} \right)^{\nu} \kappa_\nu \left(\sqrt{2\,\nu}\, \frac{|x - x'|}{\ell} \right)$$

where $\nu \geq 1/2$ is the shape parameter, ℓ is the scale parameter, $\Gamma(\cdot)$ is the Gamma function and $\kappa_\nu(\cdot)$ is the modified Bessel function of the second kind

Properties

The values $\nu = 3/2$ and $\nu = 5/2$ are usually used $\left(h = \dfrac{|x - x'|}{\ell} \right)$:

$$R_1(h;\, \nu = 3/2) = (1 + \sqrt{3}\, h) \exp(-\sqrt{3}\, h)$$

$$R_1(h;\, \nu = 5/2) = (1 + \sqrt{5}\, h + \frac{5}{3}\, h^2) \exp(-\sqrt{5}\, h)$$

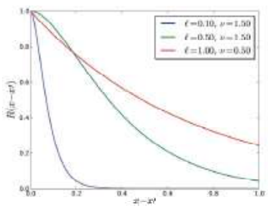
Matérn autocorrelation function

Parameter ν controls the regularity (smoothness) of the trajectories

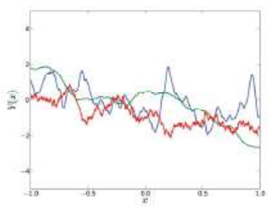
- The trajectories of such a process are $\lfloor \nu \rfloor$ times differentiable:

$$\begin{aligned} \nu = 1/2 \quad &: \quad \mathcal{C}^0 \text{ (continuous, non differentiable)} \\ \nu = 3/2 \quad &: \quad \mathcal{C}^1 \\ \nu = 5/2 \quad &: \quad \mathcal{C}^2 \end{aligned}$$

- When $\nu \to +\infty$, $R_1(h;\, \nu)$ tends to the square exponential autocorrelation



Autocorrelation function



Trajectories

Two approaches to Kriging

Data

- Given is an experimental design $\mathcal{X} = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$ and the output of the computational model $\boldsymbol{y} = \{y_1 = \mathcal{M}(\boldsymbol{x}_1), \dots, y_N = \mathcal{M}(\boldsymbol{x}_N)\}$
- We assume that $\mathcal{M}(\boldsymbol{x})$ is a realization of a Gaussian process $Y(\boldsymbol{x})$ such that the values $y_i = \mathcal{M}(\boldsymbol{x}_i)$ are known at the various points $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$
- Of interest is the prediction at a new point $\boldsymbol{x}_0 \in \mathbb{X}$, denoted by $\hat{Y}_0 \equiv \hat{Y}(\boldsymbol{x}_0, \omega)$, which will be used as a surrogate $\tilde{\mathcal{M}}(\boldsymbol{x}_0)$

$\hat{Y}_0$ is obtained as as a conditional Gaussian variable:

$$\hat{Y}_0 = Y\left(\boldsymbol{x}_0 \mid Y(\boldsymbol{x}_1) = y_1, \dots, Y(\boldsymbol{x}_N) = y_N\right)$$

Joint distribution of the predictor / observations

- For each point $\boldsymbol{x}_i \in \mathcal{X}$, $Y_i \equiv Y(\boldsymbol{x}_i)$ is a Gaussian variable:

$$Y_i = \sum_{j=1}^{p} \beta_j\, f_j(\boldsymbol{x}_i) + \sigma Z_i = \boldsymbol{f}_i^{\mathsf{T}} \cdot \boldsymbol{\beta} + \sigma\, Z_i \qquad Z_i \sim \mathcal{N}(0,1)$$

- The joint distribution of $\{Y_0, Y_1, \dots, Y_N\}^{\mathsf{T}}$ is Gaussian:

$$\left\{ \begin{array}{c} Y_0 \\ \boldsymbol{Y} \end{array} \right\} \sim \mathcal{N}_{1+N}\left(\left\{ \begin{array}{c} \boldsymbol{f}_0^{\mathsf{T}}\,\boldsymbol{\beta} \\ \mathbf{F}\,\boldsymbol{\beta} \end{array} \right\}, \sigma^2 \left[\begin{array}{cc} 1 & \boldsymbol{r}_0^{\mathsf{T}} \\ \boldsymbol{r}_0 & \mathbf{R} \end{array} \right] \right)$$

- Regression matrix $\mathbf{F}$ of size $(N \times p)$

$$\mathbf{F}_{ij} = f_j(\boldsymbol{x}_i) \quad i = 1, \dots, N,\ j = 1, \dots, p$$

- Vector of regressors $\boldsymbol{f}_0$ of size p

$$\boldsymbol{f}_0 = \{f_1(\boldsymbol{x}_0), \dots, f_p(\boldsymbol{x}_0)\}$$

- Correlation matrix $\mathbf{R}$ of size $(N \times N)$

$$\mathbf{R}_{ij} = R(\boldsymbol{x}_i, \boldsymbol{x}_j; \boldsymbol{\theta})$$

- Cross-correlation vector $\boldsymbol{r}_0$ of size N

$$\boldsymbol{r}_{0i} = R(\boldsymbol{x}_i, \boldsymbol{x}_0; \boldsymbol{\theta})$$

Mean predictor

The conditional distribution of $\widehat{Y}_0$ given the observations $\{Y(\boldsymbol{x}_i) = y_i\}_{i=1}^{n}$ is a Gaussian variable: Santner, William & Notz (2003)

$$\widehat{Y}_0 \sim \mathcal{N}(\mu_{\widehat{Y}_0}, \sigma^2_{\widehat{Y}_0})$$

Surrogate model: mean predictor

$$\mu_{\widehat{Y}_0} = \boldsymbol{f}_0^{\mathsf{T}}\,\widehat{\boldsymbol{\beta}} + \boldsymbol{r}_0^{\mathsf{T}}\mathbf{R}^{-1}\left(\boldsymbol{y} - \mathbf{F}\,\widehat{\boldsymbol{\beta}}\right)$$

where the regression coefficients $\widehat{\boldsymbol{\beta}}$ are obtained from the generalized least-square solution:

$$\widehat{\boldsymbol{\beta}} = \left(\mathbf{F}^{\mathsf{T}}\,\mathbf{R}^{-1}\,\mathbf{F}\right)^{-1}\mathbf{F}^{\mathsf{T}}\,\mathbf{R}^{-1}\,\boldsymbol{y}$$

Properties

- The mean predictor has a regression part $\boldsymbol{f}_0^{\mathsf{T}}\,\widehat{\boldsymbol{\beta}} = \sum_{j=1}^{p} \widehat{\beta}_j\, f_j(\boldsymbol{x}_0)$ and a local correction
- It interpolates the experimental design:

$$\mu_{\widehat{Y}_i} \equiv \mu_{\widehat{Y}(\boldsymbol{x}_i)} = y_i \qquad \forall \boldsymbol{x}_i \in \mathcal{X}$$

Kriging variance

- The Kriging variance reads:

$$\sigma^2_{\widehat{Y}_0} = \mathbb{E}\left[(\widehat{Y}_0 - Y_0)^2\right] = \sigma^2\left(1 - \boldsymbol{r}_0^{\mathsf{T}}\,\mathbf{R}^{-1}\,\boldsymbol{r}_0 + \boldsymbol{u}_0^{\mathsf{T}}\,\left(\mathbf{F}^{\mathsf{T}}\,\mathbf{R}^{-1}\,\mathbf{F}\right)^{-1}\,\boldsymbol{u}_0\right)$$

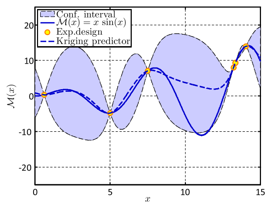
with $\boldsymbol{u}_0 = \mathbf{F}^{\mathsf{T}}\,\mathbf{R}^{-1}\,\boldsymbol{r}_0 - \boldsymbol{f}_0$

- It is made of two parts:
  - $\sigma^2\left(1 - \boldsymbol{r}_0^{\mathsf{T}}\,\mathbf{R}^{-1}\,\boldsymbol{r}_0\right)$ corresponds to the simple Kriging (when the trend is known)
  - the rest corresponds to the uncertainty due to the estimation of β from the data
- The predictor is interpolating the data in the experimental design:

$$\sigma^2_{\widehat{Y}_i} \equiv \sigma^2_{\widehat{Y}(\boldsymbol{x}_i)} = 0 \qquad \forall \boldsymbol{x}_i \in \mathcal{X}$$

Confidence intervals

- Due to Gaussianity of the predictor $\widehat{Y}_0 \sim \mathcal{N}(\mu_{\widehat{Y}_0}, \sigma^2_{\widehat{Y}_0})$, one can derive confidence intervals on the prediction
- With confidence level $(1-\alpha)$, *e.g.* 95%, one gets:



$$\mu_{\widehat{Y}_0} - 1.96\,\sigma_{\widehat{Y}_0} \le \mathcal{M}(\boldsymbol{x}_0) \le \mu_{\widehat{Y}_0} + 1.96\,\sigma_{\widehat{Y}_0}$$

- The Kriging predictor is asymptotically consistent:

$$\lim_{N \to \infty} \mathbb{E}\left[\left(\widehat{Y}_0 - Y_0\right)^2\right] = 0$$

when the size of the experimental design N tends to infinity

Outline

Introduction

So far:

- The best linear unbiased estimator assumes that the autocovariance function $\sigma^2 \, R(\boldsymbol{x}, \boldsymbol{x}'; \boldsymbol{\theta})$ is known

In practice:

- A choice is made for the family of autocorrelation function used, *e.g.* Gaussian, exponential, Matérn-ν, etc.
- The parameters of the covariance function, denoted by $\left(\sigma^2, \boldsymbol{\theta}\right)$, must be estimated from the data, *i.e.* the experimental design:

$$\mathcal{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \qquad \boldsymbol{y} = \{y_1 = \mathcal{M}(\boldsymbol{x}_1), \ldots, y_N = \mathcal{M}(\boldsymbol{x}_N)\}$$

Maximum likelihood estimation

Maximum likelihood estimation in Kriging

- Assuming that data follows a joint Gaussian distribution $\boldsymbol{Y} \sim \mathcal{N}_N(\mathbf{F}\beta, \mathbf{R}(\boldsymbol{\theta}))$ the negative log-likelihood reads:

$$-\log \mathsf{L}\left(\beta, \sigma^2, \boldsymbol{\theta} \mid \boldsymbol{y}\right) = \frac{1}{2\,\sigma^2}\,(\boldsymbol{y} - \mathbf{F}\,\beta)^{\mathsf{T}}\,\mathbf{R}(\boldsymbol{\theta})^{-1}\,(\boldsymbol{y} - \mathbf{F}\,\beta) + \frac{N}{2}\,\log(2\,\pi) + \frac{N}{2}\,\log\left(\sigma^2\right) + \frac{1}{2}\,\log\left(\det \mathbf{R}(\boldsymbol{\theta})\right)$$

Solution:

$$\widehat{\beta}(\boldsymbol{\theta}) = \left(\mathbf{F}^{\mathsf{T}}\,\mathbf{R}(\boldsymbol{\theta})^{-1}\,\mathbf{F}\right)^{-1}\mathbf{F}^{\mathsf{T}}\,\mathbf{R}(\boldsymbol{\theta})^{-1}\,\boldsymbol{y}$$

$$\widehat{\sigma^2}(\boldsymbol{\theta}) = \frac{1}{N}\,(\boldsymbol{y} - \mathbf{F}\cdot\widehat{\beta})^{\mathsf{T}}\,\mathbf{R}(\boldsymbol{\theta})^{-1}\cdot(\boldsymbol{y} - \mathbf{F}\,\widehat{\beta})$$

- Minimizing $(-\log \mathsf{L})$ is equivalent to minimizing the reduced likelihood function

$$\psi(\boldsymbol{\theta}) = \widehat{\sigma^2}(\boldsymbol{\theta})\,\det \mathbf{R}(\boldsymbol{\theta})^{1/N}$$

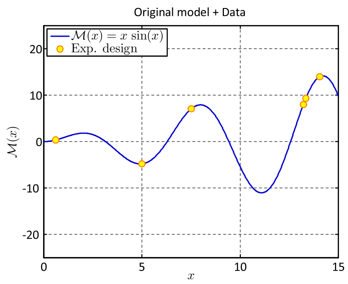
One-dimensional example

Computational model

$$x \mapsto x \sin x \qquad \text{for} \quad x \in [0, 15]$$

Experimental design

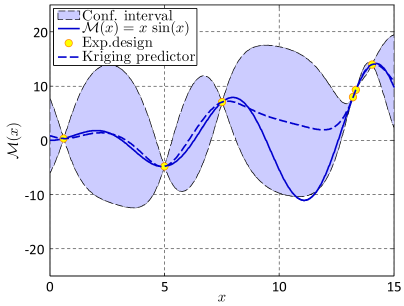
Six points selected in the range $[0, 15]$ using Monte Carlo simulation:



$$\mathcal{X} = \{0.6042 \quad 4.9958 \quad 7.5107 \quad 13.2154 \quad 13.3407 \quad 14.0439\}$$

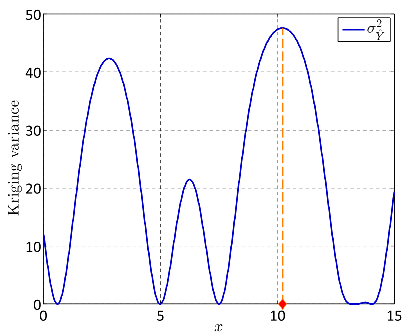
Kriging predictor

```
Trend.Type = 'ordinary' ;              % Ordinary Kriging
Covariance.Type = 'matern-5_2';        % Matérn 5/2
EstimMethod = 'ML';                    % Maximum likelihood
Optim.Method = 'BFGS';                 % BFGS algorithm
```

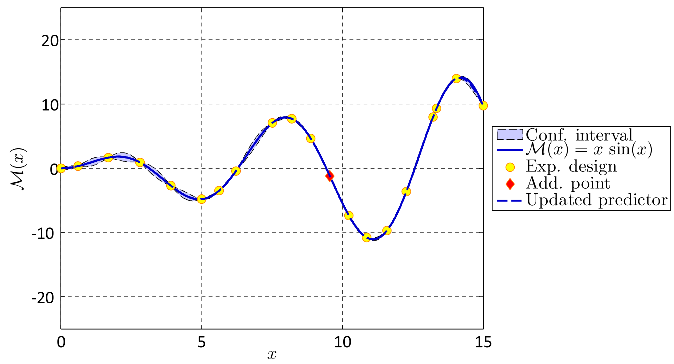


Effect of the experimental design

- In an adaptive set up, it is of interest to add points to the experimental design in regions where the Kriging variance is large



Sequential updating



Outline

Use of Kriging for structural reliability analysis

- From a given experimental design $\mathcal{X} = \left\{\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}\right\}$, Kriging yields a mean predictor $\mu_{\hat{g}}(\boldsymbol{x})$ and the Kriging variance $\sigma_{\hat{g}}(\boldsymbol{x})$ of the limit state function g
- The mean predictor is substituted for the "true" limit state function, defining the surrogate failure domain

$$\mathcal{D}_f{}^0 = \{\boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}} \, : \, \mu_{\hat{g}}(\boldsymbol{x}) \leq 0\}$$

- The probability of failure is approximated by: Kaymaz, Struc. Safety (2005)

$$P_f^0 = \mathbb{P}\left[\mu_{\hat{g}}(\boldsymbol{X}) \leq 0\right] = \int_{\mathcal{D}_f^0} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} = \mathbb{E}\left[\mathbf{1}_{\mathcal{D}_f^0}(\boldsymbol{X})\right]$$

- Monte Carlo simulation can be used on the surrogate model:

$$\widehat{P_f^0} = \frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{\mathcal{D}_f^0}(\boldsymbol{x}_k)$$

Confidence bounds on the probability of failure

Shifted failure domains

Dubourg *et al.*, Struct. Mult. Opt. (2011)

- Let us define a confidence level $(1-\alpha)$ and $k_{1-\alpha} = \Phi^{-1}(1-\alpha/2)$, *i.e.* 1.96 if $1-\alpha = 95\%$, and:

$$\mathcal{D}_f^- = \{\boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}} \, : \, \mu_{\hat{g}}(\boldsymbol{x}) + k_{1-\alpha}\, \sigma_{\hat{g}}(\boldsymbol{x}) \leq 0\}$$

$$\mathcal{D}_f^+ = \{\boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}} \, : \, \mu_{\hat{g}}(\boldsymbol{x}) - k_{1-\alpha}\, \sigma_{\hat{g}}(\boldsymbol{x}) \leq 0\}$$

- Interpretation ($1-\alpha = 95\%$):
  - If $\boldsymbol{x} \in \mathcal{D}_f^0$ it belongs to the true failure domain with a 50% chance
  - If $\boldsymbol{x} \in \mathcal{D}_f^+$ it belongs to the true failure domain with 95% chance: conservative estimation

Bounds on the probability of failure

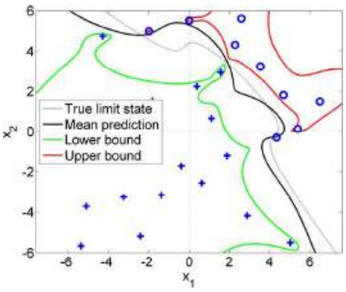
$$\mathcal{D}_f^- \subset \mathcal{D}_f^0 \subset \mathcal{D}_f^+ \qquad \Leftrightarrow \qquad P_f^- \leq P_f^0 \leq P_f^+$$

Example: hat function

Problem statement

$$g(\boldsymbol{x}) = 20 - (x_1 - x_2)^2 - 8\,(x_1 + x_2 - 4)^3$$

where $X_1,\ X_2 \sim \mathcal{N}(0,1)$



- Ref. solution:

$$P_f = 1.07 \cdot 10^{-4}$$

- Kriging surrogate:

$$P_f^- = 7.70 \cdot 10^{-6}$$
$$P_f^0 = 4.43 \cdot 10^{-4}$$
$$P_f^+ = 5.52 \cdot 10^{-2}$$

How to improve the results?

Heuristics

- The Monte Carlo estimate of P_f reads:

$$\widehat{P_f} = \frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{\mathcal{D}_f}(\boldsymbol{x}_k) \approx \frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{\mathcal{D}_f^0}(\boldsymbol{x}_k)$$

- The Kriging-based prediction is accurate when:

$$\mathbf{1}_{\mathcal{D}_f^0}(\boldsymbol{x}_k) = \mathbf{1}_{\mathcal{D}_f}(\boldsymbol{x}_k) \quad \text{for almost all } \boldsymbol{x}_k$$

i.e. if $\mu_{\hat{g}}(\boldsymbol{x})$ is of the same sign as $g(\boldsymbol{x})$ for almost all sample points

Ensure that the mean predictor $\mu_{\hat{g}}(\boldsymbol{x})$ classifies properly the MCS samples according to the sign of $g(\boldsymbol{x})$

Adaptive Kriging for structural reliability

Procedure

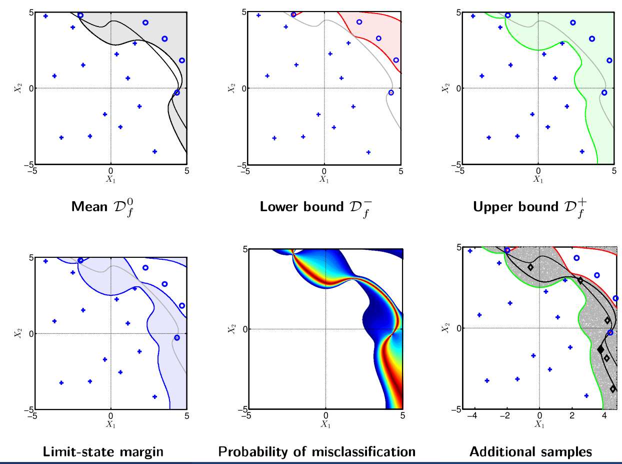
- Start from an initial experimental design $\mathcal{X}$ and a Kriging surrogate
- At each iteration:
  - Select the next point(s) to be added to $\mathcal{X}$: enrichment criterion
  - Update the Kriging surrogate
  - Compute an estimation of P_f and bounds
  - Check convergence

Adaptive Kriging for reliability analysis

Algorithm 1: Adaptive Kriging for reliability analysis

Initialization
  Initial experimental design $\mathcal{ED} = \{\chi^{(1)}, \ldots, \chi^{(n)}\}$
  Monte Carlo sample $\mathcal{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$
while *NotConverged* **do**
  Train a Kriging model $\widehat{\mathcal{M}}$ on the current experimental design
  Compute the probability of failure $\hat{P}_f^0$, and its bounds $[\hat{P}_f^-, \hat{P}_f^+]$ using $\widehat{\mathcal{M}}$
  if $(\hat{P}_f^+ - \hat{P}_f^-)/\hat{P}_f^0 \leq TOL$ **then**
    NotConverged = FALSE
  else
    Evaluate the learning function LF on $\mathcal{X}$
    Compute the next ED point: $\chi^* = \arg\min_{\boldsymbol{x}\in\mathcal{X}} LF(\boldsymbol{x})$
    Update the experimental design: $\mathcal{ED} \leftarrow \mathcal{ED} \cup \{\chi^*\}$
  end
end
Return Probability of failure $\hat{P}_f^0$ and confidence interval $[\hat{P}_f^-, \hat{P}_f^+]$

Example: hat function



Different enrichment criteria

Requirements

- It shall be based on the available information: $(\mu_{\hat{g}}(\boldsymbol{x}), \sigma_{\hat{g}}(\boldsymbol{x}))$
- It shall favor new points in the vicinity of the limit state surface
- If possible, it shall yield the best K points when distributed computing is available

Different enrichment criteria

- Margin indicator function — Ph.D Deheeger (2008); Bourinet *et al.* , Struc. Safety (2011)
- Margin classification function — Ph.D Dubourg (2011); Dubourg *et al.* , PEM (2013)
- Learning function U — Ph.D Échard (2012); Échard & Gayton, RESS (2011)
- Expected feasibility function — Bichon *et al.* , AIAA (2008); RESS (2011)
- Stepwise uncertainty reduction (SUR) — Bect *et al.* , Stat. Comput. (2012)

Learning function $U(\boldsymbol{x})$

Definition

- The learning function U is defined by: — Échard *et al.* (2011)

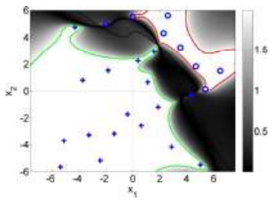
$$U(\boldsymbol{x}) = \frac{|\mu_{\hat{g}}(\boldsymbol{x})|}{\sigma_{\hat{g}}(\boldsymbol{x})}$$

Interpretation

- It describes the distance of the mean predictor $\mu_{\hat{g}}$ to zero in terms of a number of Kriging standard deviations $\sigma_{\hat{g}}$
- A small value of $U(\boldsymbol{x})$ means that:
  - $\mu_{\hat{g}}(\boldsymbol{x}) \approx 0$: $\boldsymbol{x}$ is close to the limit state surface
  - and / or $\sigma_{\hat{g}}(\boldsymbol{x}) >> 0$: the uncertainty in the prediction at point $\boldsymbol{x}$ is large
- The probability of misclassification of a point $\boldsymbol{x}$ is equal to $\Phi(-U(\boldsymbol{x}))$

Bect *et al.* , Stat. Comput. (2012)

Comparison of the enrichment criteria



Learning function U

Optimization of the enrichment criterion

$$\boldsymbol{x}_U^* = \arg\min_{\boldsymbol{x}\in\mathcal{D}_{\boldsymbol{X}}} U(\boldsymbol{x})$$

Requires to solve a complex optimization problem in each iteration

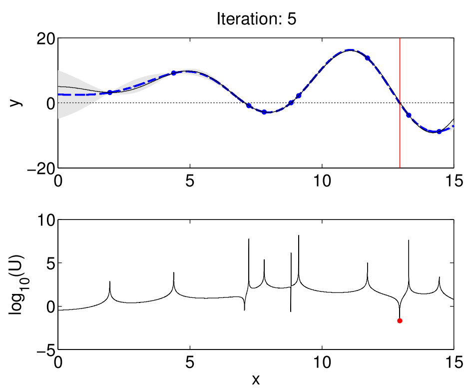
Discrete optimization over a large Monte Carlo sample $\mathfrak{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$

$$\boldsymbol{x}_U^* = \arg\min_{i=1,\ldots,n} \{U(\boldsymbol{x}_1), \ldots, U(\boldsymbol{x}_n)\}$$

Echard, B., Gayton, N. & Lemaire, M. AK-MCS: an active learning reliability method combining Kriging and Monte Carlo simulation, Structural Safety (2011)

1D Application example - U function

Limit state function: $g(x) = 5 - x \sin x$



PC-Kriging

Schöbi & Sudret, IJUQ (2015); Kersaudy *et al.* , J. Comp. Phys (2015)

Heuristics: Combine polynomial chaos expansions (PCE) and Kriging

- PCE approximates the global behaviour of the computational model
- Kriging allows for local interpolation and provides a local error estimate

Universal Kriging model with a sparse PC expansion as a trend

$$\mathcal{M}(\boldsymbol{x}) \approx \mathcal{M}^{(\mathrm{PCK})}(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} a_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}(\boldsymbol{x}) + \sigma^2 Z(\boldsymbol{x}, \omega)$$

PC-Kriging calibration

- Sequential PC-Kriging: least-angle regression (LAR) detects a sparse basis, then PCE coefficients are calibrated together with the auto-correlation parameters
- Optimized PC-Kriging: universal Kriging models are calibrated at each step of LAR

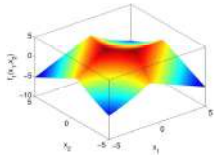
Series system

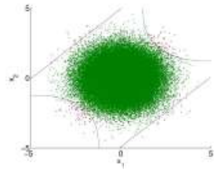
Schöbi *et al.* , ASCE J. Risk Unc. (2016)

Consider the system reliability analysis defined by:

$$g(\boldsymbol{x}) = \min \begin{pmatrix} 3 + 0.1\,(x_1 - x_2)^2 - \frac{x_1 + x_2}{\sqrt{2}} \\ 3 + 0.1\,(x_1 - x_2)^2 + \frac{x_1 + x_2}{\sqrt{2}} \\ (x_1 - x_2) + \frac{6}{\sqrt{2}} \\ (x_2 - x_1) + \frac{6}{\sqrt{2}} \end{pmatrix}$$

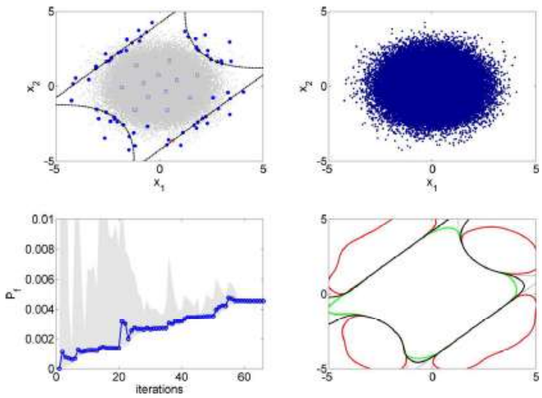
where $X_1, X_2 \sim \mathcal{N}(0, 1)$



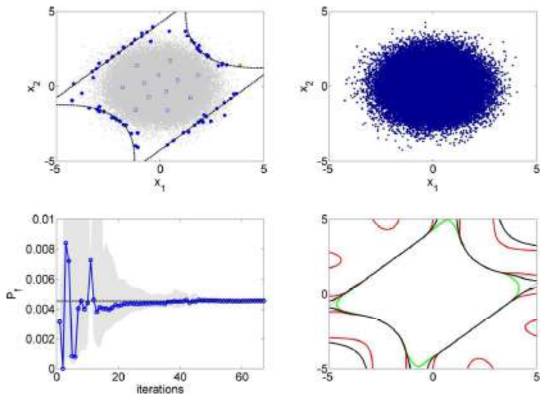


- Initial design: LHS of size 12 (transformed into the standard normal space)
- In each iteration, one point is added (maximize the probability of missclassification)
- The mean predictor $\mu_{\widehat{\mathcal{M}}}(\boldsymbol{x})$ is used, as well as the bounds $\mu_{\widehat{\mathcal{M}}}(\boldsymbol{x}) \pm 2\sigma_{\widehat{\mathcal{M}}}(\boldsymbol{x})$ so as to get bounds on P_f: $\hat{P}_f^- \leq \hat{P}_f^0 \leq \hat{P}_f^+$

Results with classical Kriging



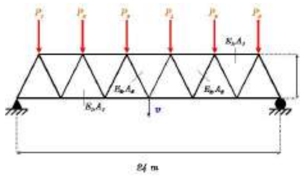
Results with PC Kriging



Outline

Elastic truss

Structural model



Blatman & Sudret (2011)

- 10 independent variables:
  - 4 describing the bars properties
  - 6 describing the loads
- Response quantity: maximum deflection U
- Reliability analysis:

$$P_f = \mathbb{P}(U \geq u_{\lim})$$

Probabilistic model

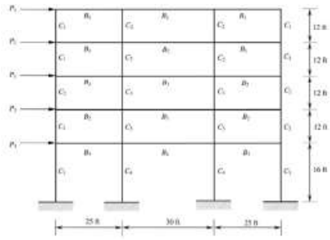
Variable	Distribution	mean	CoV
Hor. bars cross section A_1 [m]	Lognormal	0.002	0.10
Oblique bars cross section A_2 [m]	Lognormal	0.001	0.10
Young's moduli E_1, E_2 [MPa]	Lognormal	210,000	0.10
Loads $P_1, \ldots, P_6$ [KN]	Gumbel	50	0.15

Results

u_{adm}	Method	Enrichment	$\widehat{P}_f$ (CoV $[P_f]$)	$\widehat{\beta}$	N_{tot}
10 cm	**MC**	-	$\mathbf{4.29 \cdot 10^{-2}}$ (0.5 %)	1.72	10^6
	FORM	-	$2.81 \cdot 10^{-2}$	1.91	251
	OK	single	$4.32 \cdot 10^{-2}$	1.71	$\mathbf{12 + 135 = 147}$
	OK	$K = 6$	$4.31 \cdot 10^{-2}$	1.72	$12 + 26 \cdot 6 = 168$
12 cm	**MC**	-	$\mathbf{1.55 \cdot 10^{-3}}$ (2.5 %)	2.96	10^6
	FORM	-	$7.57 \cdot 10^{-4}$	3.17	236
	OK	single	$1.53 \cdot 10^{-3}$	2.96	$\mathbf{12 + 164 = 176}$
	OK	$K = 6$	$1.53 \cdot 10^{-3}$	2.96	$12 + 27 \cdot 6 = 174$
14 cm	**MC**	-	$\mathbf{3.6 \cdot 10^{-5}}$ (16.7 %)	3.97	10^6
	FORM	-	$1.29 \cdot 10^{-5}$	4.21	231
	OK	single	$3.7 \cdot 10^{-5}$	3.96	$\mathbf{12 + 110 = 122}$
	OK	$K = 6$	$3.4 \cdot 10^{-5}$	3.99	$12 + 27 \cdot 6 = 174$

Frame structure

Structural model



Blatman & Sudret, PEM, (2010)

Variable	Distribution	Mean*	Standard deviation*
P_1 (kN)	Lognormal	133.454	40.04
P_2 (kN)	"	88.97	35.59
P_3 (kN)	"	71.175	28.47
E_4 (kN/m²)	Truncated Gaussian over [0, +∞)	2.1738×10^7	1.9152×10^6
E_5 (kN/m²)	"	2.3796×10^7	1.9152×10^6
I_6 (m⁴)		8.1344×10^{-3}	1.0834×10^{-3}
I_7 (m⁴)		1.1509×10^{-2}	1.2980×10^{-3}
I_8 (m⁴)		2.1375×10^{-2}	2.5961×10^{-3}
I_9 (m⁴)		2.5961×10^{-2}	3.0288×10^{-3}
I_{10} (m⁴)	"	1.0812×10^{-2}	2.5961×10^{-3}
I_{11} (m⁴)		1.4105×10^{-2}	3.4615×10^{-3}
I_{12} (m⁴)		2.3279×10^{-2}	5.6249×10^{-3}
I_{13} (m⁴)		2.5961×10^{-2}	6.4902×10^{-3}
A_{14} (m²)		3.1256×10^{-1}	5.5815×10^{-2}
A_{15} (m²)		3.7210×10^{-1}	7.4420×10^{-2}
A_{16} (m²)		5.0606×10^{-1}	9.3025×10^{-2}
A_{17} (m²)		5.5815×10^{-1}	1.1163×10^{-1}
A_{18} (m²)	"	2.5302×10^{-1}	9.3025×10^{-2}
A_{19} (m²)		2.9117×10^{-1}	1.0232×10^{-1}
A_{20} (m²)		3.7303×10^{-1}	1.2093×10^{-1}
A_{21} (m²)		4.1860×10^{-1}	1.9537×10^{-1}

* The mean value and standard deviation of the cross sections, moments of inertia and Young's moduli are those of the untruncated Gaussian distributions.

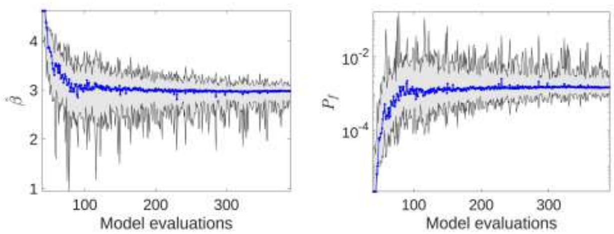
Probabilistic model

- 21 correlated variables (3 loads, 2 Young's moduli, 8 cross-section properties) using a Gaussian copula (Nataf transform)
- Reliability analysis (max. horizontal displacement):

$$P_f = \mathbb{P}(U \geq u_{\lim}) \qquad u_{\lim} = 5 \text{ cm}$$

Results

u_{adm}	Method	Enrichment	$\widehat{P}_f$ (CoV $[P_f]$)	$\widehat{\beta}$	N_{tot}
5 cm	Ref.	-	$1.54 \cdot 10^{-3}$ (1 %)	2.96	41'941
	FORM	-	$1.01 \cdot 10^{-3}$ (-)	3.08	241
	OK	single	$1.48 \cdot 10^{-3}$ (3.7 %)	2.97	390

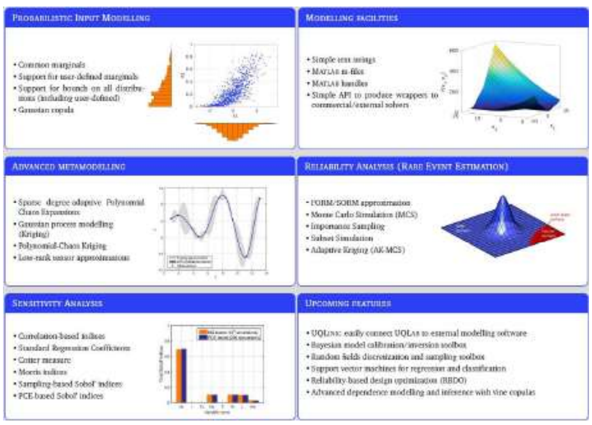


Conclusions

- Estimating low probabilities of failure in high-dimensional problems requires more refined algorithms than plain MCS
- Recent research on surrogate models (*e.g.* Kriging) and active learning has brought new extremely efficient algorithms
- Active learning has also been recently developed using bootstrap using polynomial chaos expansions as surrogates. Marelli & Sudret, ICASP (2017); Struc. Safety (2018)
- Accurate estimations of P_f's (not of β !) are obtained with $\mathcal{O}(100)$ runs of the computer code independently of their magnitude
- All the presented algorithms are available in the general-purpose uncertainty quantification software UQLab

UQLab

www.uqlab.com



UQLab: The Uncertainty Quantification Laboratory

http://www.uqlab.com

- Release of V0.9 on July 1st, 2015; V0.92 on March 1st, 2016
- Release of V1.0 on April 28th, 2017 UQLabCore + Modules
- 1140 licences granted, 670 active, 57 countries
- Presentations at summer schools in Germany (Weimar, Berlin, Magdeburg) in summer 2016 and 2017, at SIAM UQ 2016, UNCECOMP 2017, etc.

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Canada	15
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UQLab users

Questions ?

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Thank you very much for your attention !

EFFICIENT MONTE CARLO ALGORITHMS FOR SOLVING RELIABILITY PROBLEMS

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Assessing risk quantitatively requires the quantification of the probability of occurrence of a specific event by properly propagating the uncertainty through the model that predicts the quantities of interest. The estimation of small probabilities of failure from computer simulations is a classical problem in engineering. In principle, rare failure events can be investigated through Monte Carlo simulation. However, this is computationally prohibitive for complex systems because it requires a large number of samples to obtain one failure sample.

Advanced Monte Carlo methods aim at estimating rare failure probabilities more effi- ciently than direct Monte Carlo. Unfortunately, high dimension and model complexity make it extremely difficult to improve the efficiency of Monte Carlo algorithms purely based on prior knowledge, leaving algorithms that adapt the generation of samples during simulation the only choice.

Importance Sampling [3], Subset Simulation [1] and Line Sampling [2] algorithms have become popular methods to solve it, thanks to its robustness in application and still savings in the number of simulations to achieve a given accuracy of estimation for rare events, compared to many other Monte Carlo approaches. Some recent advancement and numerical implementation [4] of these algorithms will be presented.

References

[1] Siu Kui Au and Edoardo Patelli. Subset simulation in finite-infinite dimensional space. Reliability Engineering & System safety, 148:66–77, 2016.

[2] Marco de Angelis, Edoardo Patelli, and Michael Beer. Advanced line sampling for efficient robust reliability analysis. Structural safety, 52:170–182, 2015.

[3] Marco de Angelis, Edoardo Patelli, and Michael Beer. Forced monte carlo simulation strategy for the design of maintenance plans with multiple inspections. ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems. Part A: Civil Engineering, page D4016001, 2016.

[4] Edoardo Patelli, Matteo Broggi, Silva Tolo, and Jonathan Sadeghi. Cossan software a multidisciplinary and collaborative software for uncertainty quantication. In 2nd International Conference on Uncertainty Quantification in Computational Sciences and Engineering, volume Eccomas Proceedia ID: 5364, pages 212–224, 2017.

COMPUTATIONAL CHALLENGES IN THE RELIABILITY ASSESSMENT of ENGINEERING STRUCTURES

Efficient Monte Carlo algorithms for solving reliability problems

Delft, The Netherlands, 26 January 2018

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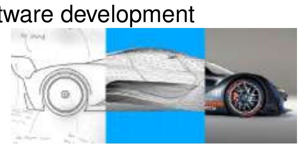


Centre for Doctorate Training (CDT)

www.liv.ac.uk/risk-and-uncertainty-cdt

Centre for Doctoral Training on Quantification and Management of Risk & Uncertainty in Complex Systems & Environments

Highlights

- 80+ students (5 cohorts)
- 36 Industrial Partners
- 5.8 Million Pounds in Funding
- Meeting the needs of industry
- Throughput of future leaders

EPSRC Pioneering research and skills

E·S·R·C ECONOMIC & SOCIAL RESEARCH COUNCIL

Outline

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Modelling and Design

Virtual (numerical) Prototypes

- Very accurate deterministic solvers
- Advanced modelling tools
- Geometry, meshing, static and dynamic analysis, fluid/structure interaction, crack propagation, ballistic impact

Risk is often misestimated

- Models are deterministic without incorporating any measure of uncertainty (Columbia accident report)
- Inadequate assessment of uncertainties, unjustified assumptions (NASA-STD-7009)
- Looking for the “black swan” (e.g. Fukushima)

Questions to be answered

- How are the uncertainties modelled?
- What is the variability of the quantities of interest?

- **How does the uncertainty affect the performance of the model/system?**
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Questions to be answered

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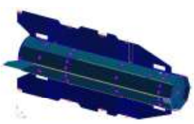
⇒ Answers by uncertainty quantification

Challenges

Computational cost of the analysis



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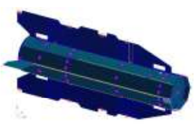

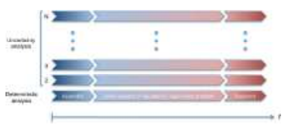

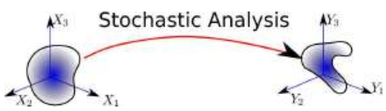

Stochastic analysis

Requirements

Efficient analysis requires:

- High Performance Computing
- Advanced simulation methods

Computational modelling is the third pillar of scientific research

Stochastic analysis

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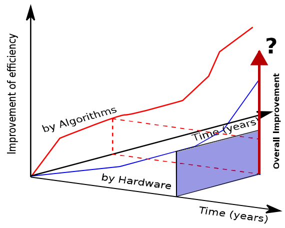


Computational modelling is the third pillar of scientific research

Reliability Analysis

The ability of a system or component to perform its required functions under stated conditions for a specified period of time.

Reliability is a probability

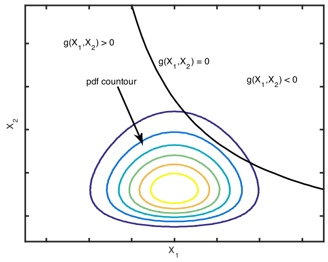
$$R(t) = Pr\{T > t\} = \int_t^\infty f(\mathbf{X})\, d\mathbf{X}$$

where $f(\mathbf{X})$ is the failure probability density function and t is the length of the period of time

Performance function $g(X_1, \cdots, X_n)$

Describe the status of the system

- Failure domain: $g \leq 0$
- Safe domain: $g \geq 0$
- Limit State Function: $g(X_1, \cdots, X_n) = 0$ ($N - 1$ dimension surface)



Model must be evaluated to determine if $\mathbf{X} \in \mathcal{F}$

Structural reliability problem

$$P_f = P(g(X_1, \cdots, X_n) \leq 0) = \int \cdots \int_{g(\mathbf{X})<0} f_{(\mathbf{X})}(\mathbf{x}) d\mathbf{x}$$

Exact solution of this integral is possible only with multivariate normal random variables and linear limit state functions

Tools

- Approximated methods (FORM, SORM,etc..)
- Monte Carlo simulation
- Important sampling, Line sampling, Subset simulation

Which tool to use?

Challenges

i. High-dimensional ($n > 30, 40$)
ii. Multiple failure modes: $P_f = P(\Phi(\mathbf{X}))$ (system reliability)
iii. Small failure probabilities: $P_f \leq 10^{-4}, 10^{-6}$

Outline

Safety Margin

Fundamental problem

For normal random variables and linear performance function

- $X_i \sim N(\mu_{x_i}, \sigma_{x_i})$
- $g(\mathbf{x}) = a_0 + \sum_{i=1}^n a_i x_i$
- $M = D - C = g(\mathbf{x})$ is called safety margin

$$M \sim N(\mu_M, \sigma_M^2)$$

(remember linear combination of normally distributed random variables)

$\mu_M = a_0 + \sum_{i=1}^n a_i \mu_{x_i} \quad \sigma_M^2 = \sum_{i=1}^n a_i^2 \sigma_{x_i}^2 + \sum_{i=1}^n \sum_{j=1, j\neq i}^n \rho_{ij} a_i a_j \sigma_i \sigma_j$

Safety Margin

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Reliability index β

Analytical method

$$\beta = \mu_M / \sigma_M$$

By transforming the variables in the standard normal space U

- Probability of failure
  $P_f = P(M < 0) = P(\mu_M - U\sigma_M \leq 0) = P\left(U \leq -\frac{\mu_M}{\sigma_M}\right)$
- $P_f = \phi(-\beta)$ where $\Phi(\cdot)$ is the CDF of a U

Geometrical interpretation

Safety index β represents the number of standard deviation by which the mean value of the safety margin M exceeds zero

Reliability index β

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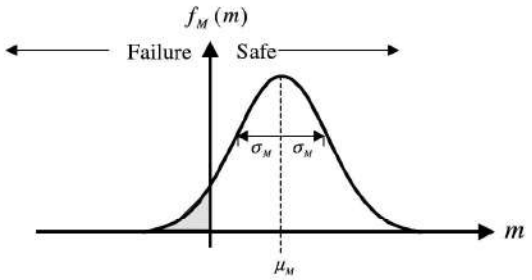
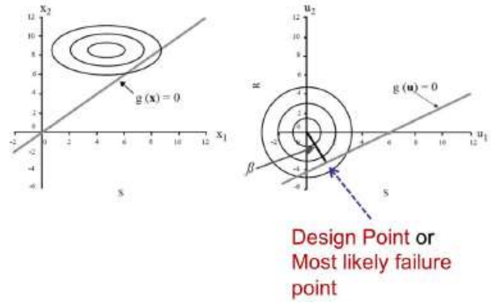
Geometrical Interpretation/1



Reliability index and Design Point

Smallest distance from the origin of the standard normal space with the limit state function



First Order Reliability Method (FORM)

Linearization in Standard Normal Space

- Transform Random Variables in Standard Normal Variables
- Identify the closed point of the limit state function to the origin (Most Probable Point)
- $\beta = min_{u \in \{g(\mathbf{u})=0\}} \sqrt{\sum_i u_i^2}$
- The distance β gives an approximate value of the probability of failure

Method proposed by Hasofer and Lind in 1974

First Order Reliability Method (FORM)

Applicability and limitation

i High-dimensional: No*!
ii Multiple failure modes: Possible+
iii Small failure probabilities: Yes

* Valdebenito, M.; Pradlwarter, H. & Schuëller, G. The Role of the Design Point for Calculating Failure Probabilities in view of Dimensionality and Structural Non Linearities, *Structural Safety*, 2010, 32, 101-111

\+ It will be explained later

Outline

Monte Carlo method

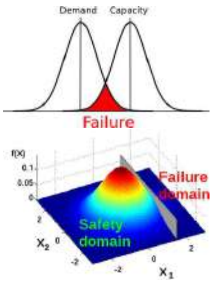
Evaluation of Definite Integrals

$$G = \int g(x)f(x)dx$$

x can be seen as a random variable;
$f(x)$ has characteristic of a probability density function $\rightarrow g(x)$ is also a random variable.

$$E[g(x)] = \int g(x)f(x)dx = G \qquad Var[g(x)] = E[g^2(x)] - G^2$$

Failure quantification



$$G = \int_{\mathcal{F}} g(\mathbf{x})d\mathbf{x} = \int_{\mathcal{F}} g(x_1, \ldots, x_n)dx_1 \ldots dx_n$$

$$\int_{\mathcal{F}} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} = \int \mathbb{I}_{\mathcal{F}}(\mathbf{x})\, f_{(\mathbf{X})}(\mathbf{x})\, d\mathbf{x}$$

where:

$$\mathbb{I}_{\mathcal{F}}(\mathbf{X}) = \begin{cases} 0 & \iff \mathbf{X} \in \mathcal{S} \\ 1 & \iff \mathbf{X} \in \mathcal{F} \end{cases}$$

Monte Carlo darts method

$$P_f = \int \mathbb{I}_{\mathcal{F}}(\mathbf{x})\, f_{(\mathbf{X})}(\mathbf{x})\, d\mathbf{x} \approx \frac{1}{N}\sum_{k=1}^{N} \mathbb{I}_{\mathcal{F}}(\mathbf{X}^{(k)})$$

- Generate sample N points $\mathbf{x_i}$ from $f_{(\mathbf{X})}(\mathbf{x})$
- Evaluate $g(\mathbf{x_i})$ (prize)
- Computed expected prize

Monte Carlo simulation

- **Always working**
- Provide the exact solution for $N \to \infty$
- Does not required any prior knowledge
- Accuracy $N \propto \frac{1}{P_f}$ (**independent of number of variables**)
- Infeasible for expensive models and low P_f

Monte Carlo simulation

Applicability and limitation

i High-dimensional: Yes
ii Multiple failure modes: Yes
iii Small failure probabilities: usually not^{+}

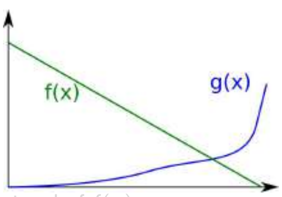
$^{+}$ Yes for "non-expensive" models (or if surrogate models are used)

Outline

Variance reduction technique

Motivation

if $f(x)$ is large when $g(x)$ is small (and vice-versa)
Large error estimator
$Var[G_N] = \frac{1}{N}\left(E\left[g^2(x)\right] - G^2\right)$



A different function $f_1(x)$ can be used instead of $f(x)$

$$G = \int_D \left[\frac{f(x)}{f_1(x)} g(x)\right] f_1(x)dx \equiv \int_D g_1(x) f_1(x) dx$$

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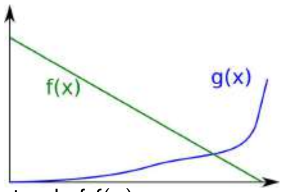

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Variance reduction technique

Monte Carlo biased dart game

$$G = \int g(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = \int \frac{g(\mathbf{x}) f(\mathbf{x})}{f_1(\mathbf{x})} f_1(\mathbf{x}) d\mathbf{x} = \int g_1(\mathbf{x}) f_1(\mathbf{x}) d\mathbf{x}$$

- Sample from $X \sim f_1(x)$
- Collect prize $g_1 = \frac{f(x)}{f_1(x)} g(x)$
- Estimate $G_{1N} = \frac{1}{N}\sum_{i=1}^{N} g_1(x_i)$

Computational methods Importance sampling

Variance reduction technique

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Importance Sampling

Advanced Monte Carlo Simulation

$$P_f = \int \mathbb{I}_{\mathcal{F}}(\mathbf{x})\, f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

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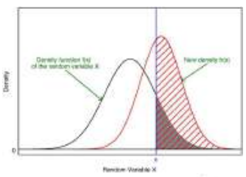
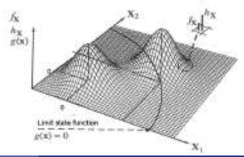
$$\hat{P}_f = \frac{1}{N}\sum_{k=1}^{N} \mathbb{I}_{\mathcal{F}}(\mathbf{X}^{(k)}) w(\mathbf{X}^{(k)})$$





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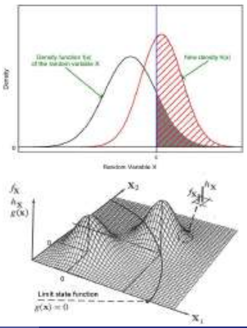

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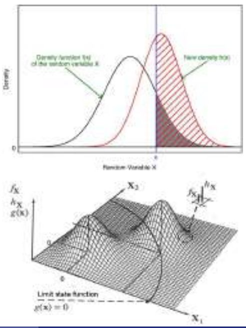


Importance sampling

Applicability and limitation

Requires prior information of the failure region

i High-dimensional: Possible but difficult +
ii Multiple failure modes: Yes *
iii Small failure probabilities: Yes

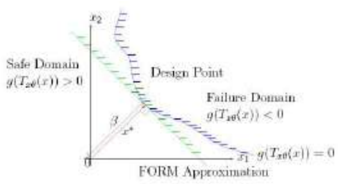
+ Difficult to define importance sampling density
* Patelli, E.; Pradlwarter, H. J. & Schuëller, G. I. On Multinormal Integrals by Importance Sampling for Parallel System Reliability *Structural Safety*, 2011, 33, 1-7
* Mahadevan, S. & Raghothamachar, P. Adaptive simulation for system reliability analysis of large structures *Computers & Structures*, 2000, 77, 725 - 734

Outline

Line sampling

Advanced Monte Carlo simulation

- Based on the linearisation of limit state function
- It can be see and a weighed average of FORM
- Areas with larger mass density contribute most



Line Sampling

Some maths

$P_F = \int_{-\infty}^{\infty} \mathbb{I}_{\mathcal{F}}(\boldsymbol{u})\ h_{\mathcal{N}}(\boldsymbol{u})\ d\boldsymbol{u}$, $h_{\mathcal{N}}(\boldsymbol{u})$ is invariant to rotation of the coordinate axes. Hence,

$$P_F = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mathbb{I}_{\mathcal{F}}(\boldsymbol{u}) \phi(u_1) du_1 \right) \prod_{i=2}^{n} \phi(u_i) du_i$$

u_1 can be interpreted as "important direction" pointing towards the failure region: $\boldsymbol{\alpha} \in \mathbb{R}^n$

$$P_F = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mathbb{I}_{\mathcal{F}}(\boldsymbol{u}) \phi(\alpha) d\alpha \right) \prod_{i=2}^{n} \phi(u_i) du_i$$

Line Sampling

Some maths (cont)

$\boldsymbol{u}^{\perp} = \{0, \boldsymbol{u}_{2:n}\}$ lies on the hyperplane orthogonal $\boldsymbol{\alpha}$.

$$w(\boldsymbol{u}^{\perp}) = \int_{-\infty}^{\infty} \mathbb{I}_{\mathcal{F}}(\boldsymbol{u})\, \phi(\alpha)\, d\alpha \approx \Phi(-|c^*|)$$

c^* smallest value of α where $\mathbb{I}_{\mathcal{F}}(\boldsymbol{u})$ steps from 0 to 1

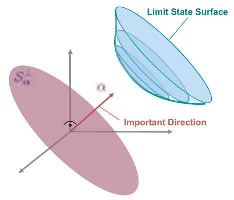
$w(\boldsymbol{u}^{\perp})$ is a measure of likelihood for the variable $\boldsymbol{u}^{\perp}$ to be in the failure domain

$$\hat{P}_f = \frac{1}{N_L} \sum_{i=1}^{N_L} \Phi(-|c^*|)$$

Line Sampling

Procedure (Working in Standard Normal Space)

- Identify direction $\boldsymbol{\alpha}$
- Samples in the hyperplane $S_{\alpha}^{\perp}$
- For each points $X^{\perp}$ generate parallel lines
- Evaluate function along lines
- Identify intersection with limit state
- Compute first order reliability for each line



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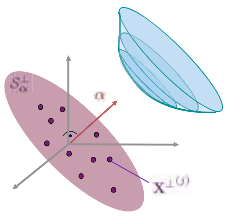

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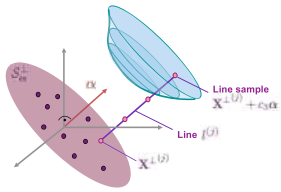

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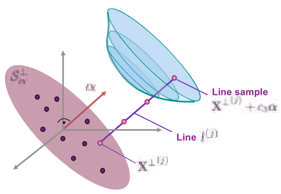

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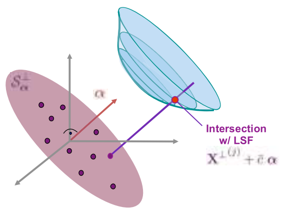

Line Sampling

Procedure (Working in Standard Normal Space)

- Identify direction α
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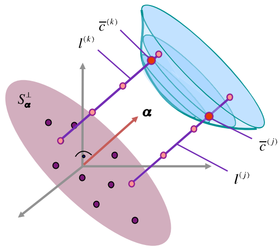
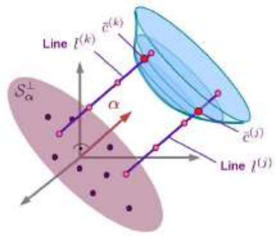


Line Sampling

Features

- Efficient approach (samples required are independent of the failure probability)
- Efficient in high dimensional space
- Requires an approximate direction pointing towards failure region
- Might not perform well with strongly non-linear performance function

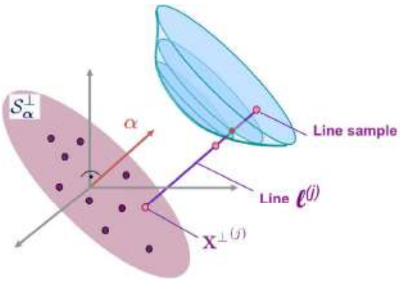


Advanced Line Sampling

Line search

Strategy

- Identify c^j using quasi Newton method
- Identify next closest line $(j+1)$
- Start line search from c^j
- Process next line



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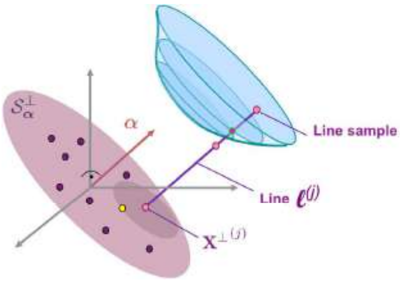

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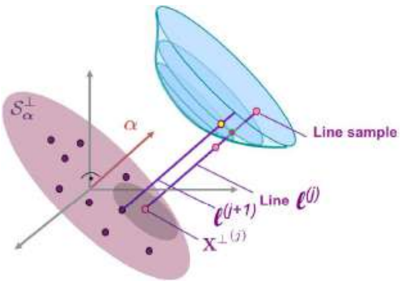

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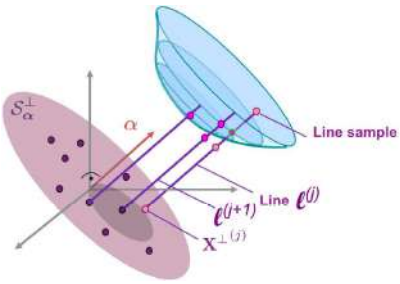
- Start line search from c^j
- Process next line

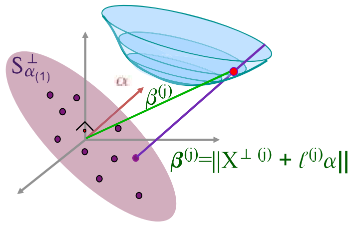


Advanced Line Sampling

Updating Importance Direction

- Update automatically the importance direction
- Recompute P_f without re-evaluating the model

Points in SNS invariant to any space rotation



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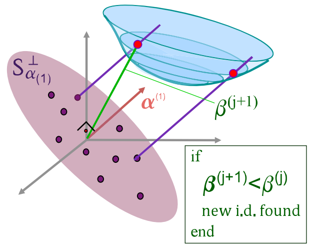

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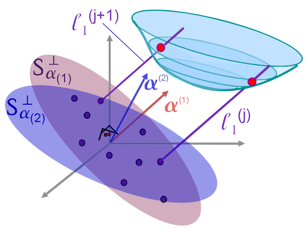

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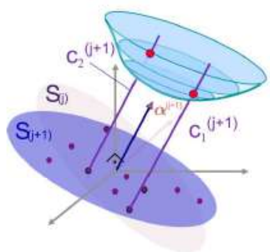
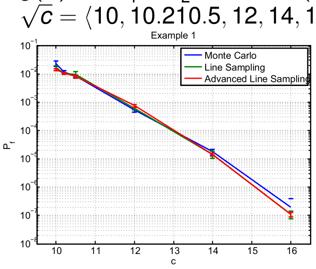
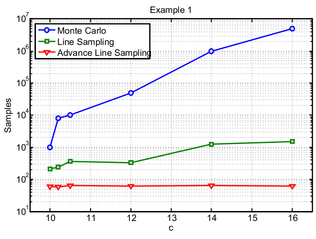


Example 1

$g(\boldsymbol{x}) = -x_1^2 + x_2^2 c \quad x_1 \sim N(5, 2^2) \quad x_2 \sim N(2, 2^2)$

$\sqrt{c} = \langle 10, 10.210.5, 12, 14, 16\rangle$





Example 2

$g(\boldsymbol{x}) = -\sum_i^N x_i^2 + c$

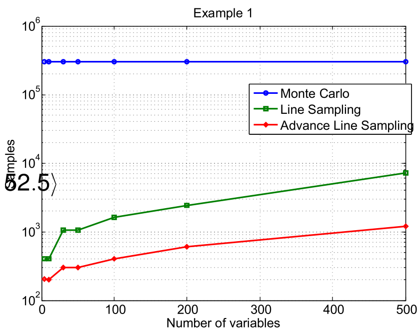
$x_i \sim N(2, 1)$

$\sqrt{c} =$

$\langle 7, 9.33, 14.7, 18.1, 24.8, 34, 52.5\rangle$

$N =$

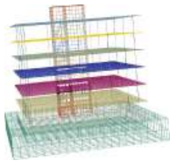
$\langle 4, 10, 30, 50, 100, 200, 500\rangle$



Multi-storey Building

- Multi-storey building modelled with ABAQUS
- 8,200 elements and 66,300 DOFs
- 244 independent uncertain quantities considered
- Aim: failure probability due to static load

Line sampling
Only **100** model evaluations
Estimated failure probability: $1.3 \cdot 10^{-5}$



Line sampling

Applicability and limitation

Requires prior information of the failure region

i High-dimensional: Yes *
ii Multiple failure modes: Possible +
iii Small failure probabilities: Yes*

* de Angelis, M.; Patelli, E. & Beer, M. An efficient strategy for interval computations in risk-based optimization ICOSSAR 2013, June 16-20, 2013
* de Angelis, M.; Patelli, E. & Beer, M. Advanced line sampling for efficient robust reliability analysis *Structural safety*, Elsevier, 2015, 52, 170-182
+ It will be explained later

Outline

Subset simulation

Compute small failure probability as a product of larger conditional probabilities

- $P(F_1)$ usually by means of plain Monte Carlo
- Identify first limit state function F_1
- Generate conditional samples from $P(F_2|F_1)$

$$\hat{P}_f = P\left(\bigcap_{i=1}^{m} F_i\right) = P(F_1)\prod_{i=1}^{m-1} P(F_{i+1}|F_i)$$

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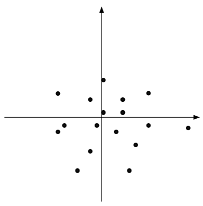

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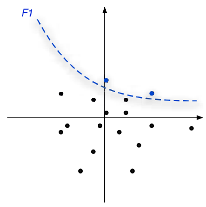

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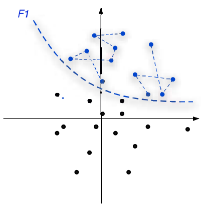

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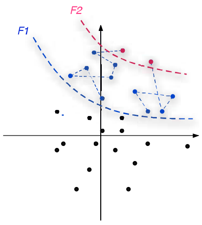

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Subset simulation

Requirements and challenges

How to generate conditional samples from $P(F_{i+1}|F_i)$

- Using Markov Chain Monte Carlo (component-wise updates Metropolis-Hastings algorithm)
  - Sample new state from a proposal distribution $\boldsymbol{X}' \leftarrow \pi(\boldsymbol{X})$
    - for each component X_i accepted with probability
  - Accepted if $\boldsymbol{X}' \in F_k$
- Require definition of proposal PDF
- Sequential approach

Subset simulation - MCMC (component-wise)

for each k-level **do**
  $\boldsymbol{X}_k \leftarrow F(\boldsymbol{x}|F_k)$
  for each component X_i **do**
    generate new component $X_i' \leftarrow \pi(X_i)$
    accept with probability $r = min(1, \phi(X_i')/\phi(X_i))$
  end for
  for each proposed candidate $\boldsymbol{X}'$ **do**
    if $\boldsymbol{X}' \in F_k$ **then**
      $\boldsymbol{X}_{(k+1)} = \boldsymbol{X}'$
    else
      $\boldsymbol{X}_{(k+1)} = \boldsymbol{X}_k$
    end if
  end for
end for

Subset simulation-∞

Equivalent problem

Subset-MCMC efficiency increases with dimensionality

Equivalent problem

- Each random variable X represented by an arbitrary (and hence possibly infinite) number of *hidden* variables $\boldsymbol{Z}$

$$X_i = \frac{1}{\sqrt{n'}}\sum_{j=1}^{n_j'} Z_{ij}$$

- Linear transformation: $\boldsymbol{X} = \boldsymbol{LZ}$ response depends on $\boldsymbol{X}$

Central Limit theorem: The sum of many IID random variables with defined expected value and finite variance will be approximately normally distributed.

Subset simulation-∞

Component-wise updates Metropolis algorithm

$\boldsymbol{X}_k \leftarrow F(\boldsymbol{x}|F_k)$
for each component X_i **do**
  for each hidden variable X_{ij} **do**
    generate new component $X_{ij}' \leftarrow \pi(X_{ij} - Z_{ij})$
    accept with probability $r = min(1, \phi(X_{ij}')/\phi(X_{ij}))$
  end for
  Set $X_i' \leftarrow \frac{1}{\sqrt{n'}}\sum_{j=1}^{n_j'} Z_{ij}$
end for

Study the limiting behaviour of the MCMC algorithm

Subset simulation-∞

for $n \to \infty$ the candidate X' is distributed as Gaussian distribution with mean aX_i and variance s_i^2:
$\kappa_i = \int_0^\infty w^2 \Phi\left(-\frac{w}{2}\right) \pi_i(w) dw$,
$s_i = 4\kappa_i - 4\kappa_i^2$, $a_i = 1 - 2\kappa_i$ $a_i^2 + s_i^2 = 1$

Papaioannou I., Betz W., Zwirglmaier K., Straub D.: MCMC algorithms for subset simulation. Probabilistic Engineering Mechanics, 2015, 41: 89-103
Siu-Kui Au and Edoardo Patelli Subset simulation in finite-infinite dimensional space. Reliability Engineering and Safety System, 2016 148 66-77

- Conditional PDF **does not depend** on hidden variables
- Allows to directly generate samples X

Computational methods | Subset simulation

Subset simulation-∞

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- Conditional PDF **does not depend** on hidden variables
- Allows to directly generate samples X

Edoardo Patelli | University of Liverpool | 26 January 2018 - 51 -

Subset simulation-∞ (Algorithm)

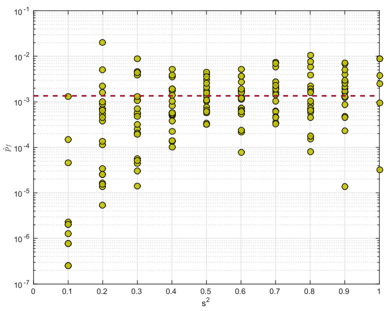
```
a ← √(1 − s²) where s = [s_1, ..., s_n] represents the vector of chosen standard deviation for each component X_i
for each SubSim k-level do
    X_k ← F(x|F_k)
    generate n candidates X' ~ N(aX^(k), s)
    for each proposed candidate X' do
        if X' ∈ F_k then
            X_(k+1) = X'
        else
            X_(k+1) = X_k
        end if
    end for
end for
```


SubSim-∞

Matlab implementation

```
% bk = threshold of the current level
% Mx = matrix of initial samples (Nvariables,NinitialSamples)
% Vstd = vector of standard deviations
Va = sqrt (1-Vstd.^2);
Mx = repmat(Mx,Nsamples,1);
MxCandidate = normrnd(Va.*Mx,Vstd);
% Evaluate the model (myModel)
Vg=myModel(MxCandidate);
% Identify accepted samples (myModel)
Vaccepted=find((Vg <= bk))==1);
Mx(Vaccepted,:)=MprososedSamples(Vaccepted,:);
```


Effect of the variance s^2



Example: Multiple failure regions

Benchmark example #3, presented in Engelund 1993

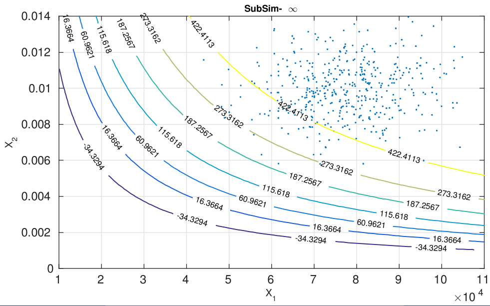
Performance function: $g_4(\mathbf{x}) = X_1 X_2 - PL$

Variable	Distribution	Mean	Std
X_1	Normal	78064.4	11709.7
X_2	Normal	0.0104	0.00156
P	Deterministic	14.614	-
L	Deterministic	10.000	-

S. Engelund, R. Rackwitz A benchmark study on importance sampling techniques in structural reliability, Structural Safety, 1993, 12(4), 255-276

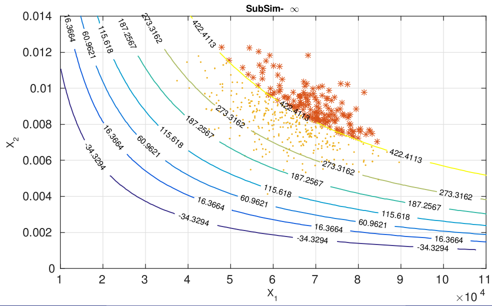
Multiple failure regions

Results SubSim-∞



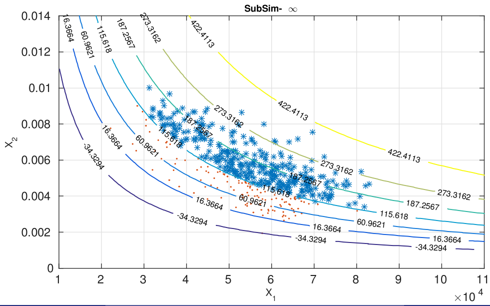
Multiple failure regions

Results SubSim-∞



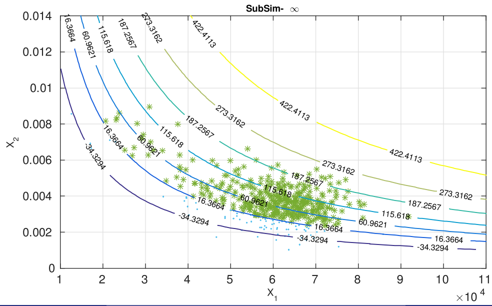
Multiple failure regions

Results SubSim-∞



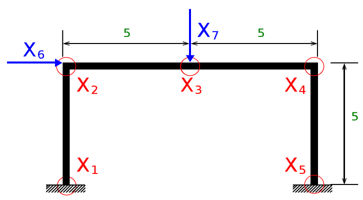
Multiple failure regions

Results SubSim-∞

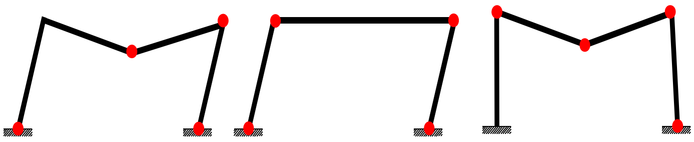


Outline

Multiple failure modes



Multiple failure modes



Multiple failure modes

Each failure mode can be analysed separately (if known)
Define separate failure events

$$p_f = \int \cdots \int_{E_1 \bigcup \cdots \bigcup E_k} f_{X_1, X_2, \ldots}(X_1, \ldots, X_n) dX_1, \cdots, dX_n$$

can be approximated using only the most significant failure sequences S_j:

$$p_f = P(\bigcup_{i=1}^{k} E_i) \approx P(\bigcup_{j=1}^{S_j} \ldots E_j)$$

Divide Et Impera

Multiple failure modes

Any failure can be reduced to a combination of parallel and series system.

$$P_f(sys) = P_A \cap P_B \cap P_C = P_A * P_B * P_C$$

$$P_f(sys) = P_A \cup P_B \cup P_C =$$
$$P_A + P_B + P_C - P_A * P_B - P_A * P_C - P_B * P_C + P_A * P_B * P_C$$

Approximate methods

- Polynomial fitting*
- Product of Conditional Marginals+
- Bounding techniques (first and second order)#

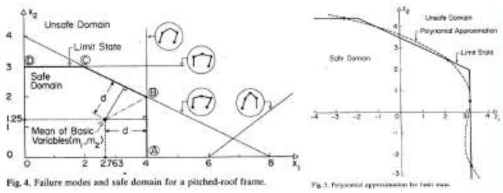


Fig. 4. Failure modes and safe domain for a pitched-roof frame.

Approximate methods

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- Product of Conditional Marginals+
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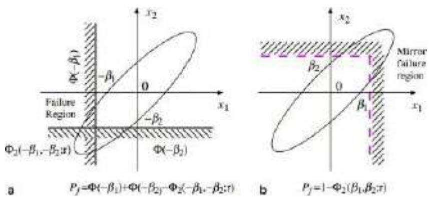


Approximate methods

- Polynomial fitting*
- Product of Conditional Marginals+
- Bounding techniques (first and second order)#

$$p_l \leq p_f \leq p_u$$

$$p_l = \sum_{i=1}^{m} \max \left[0, P(E_i) - \sum_{j=1}^{i-1} P(E_i \cap E_j) \right]$$

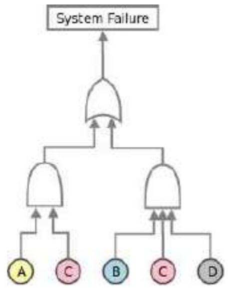
$$p_u = \sum_{i=1}^{m} P(E_i) - \sum_{i=2}^{m} \max_{j<i} P(E_i \cap E_j)$$

Approximate methods

- Polynomial fitting*
- Product of Conditional Marginals+
- Bounding techniques (first and second order)#

* Grigoriu, M. Methods for approximate reliability analysis. *Structural Safety*, 1982, 1, 155-165
+ Yuan, X.-X. & Pandey, M. Analysis of approximations for multinormal integration in system reliability computation, *Structural Safety*, 2006, 28, 361 - 377
# Ditlevsen, O. Narrow Reliability Bounds for Structural Systems *Mechanics Based Design of Structures and Machines*, 1979, 7, 453-472

Simulation methods

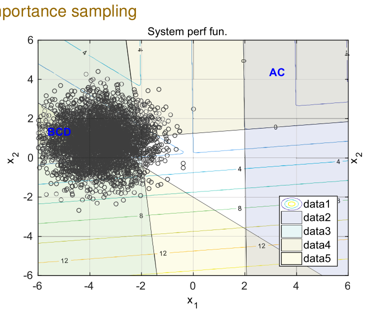


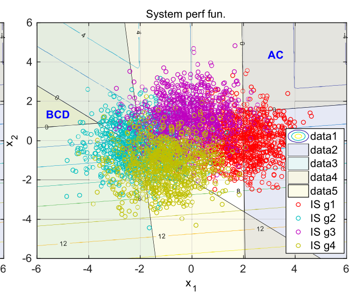
- Simulation methods can be used on estimate basic events (based on diffent performance functions)
- Combine the results to estimate the top event

Only Monte Carlo sampling guarantees the identification of all the failure modes!

Simulation methods

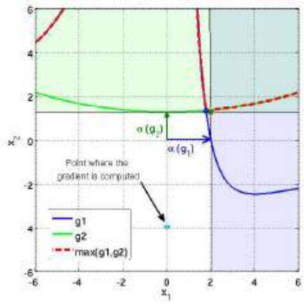
Importance sampling





Efficient Importance sampling

- Compute the design point of the intersection of two events (iteratively)
- Construct an important sampling density around the desing point



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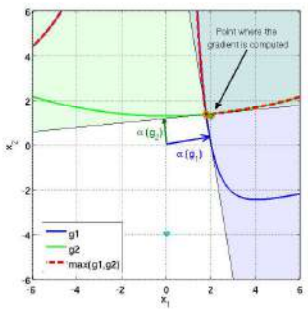
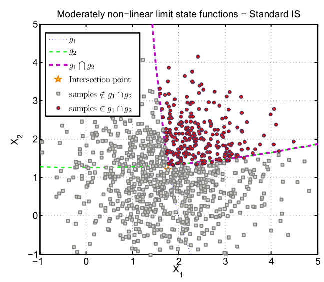


Efficient Importance sampling

Parallel system

- Create an importance denitity centered on the Desing Point
- Generate samples mostly (only) in the failure region*.



*Patelli, E.; Pradlwarter, H. J. & Schuëller, G. I. On Multinormal Integrals by Importance Sampling for Parallel System Reliability *Structural Safety*, 2011, 33, 1-7

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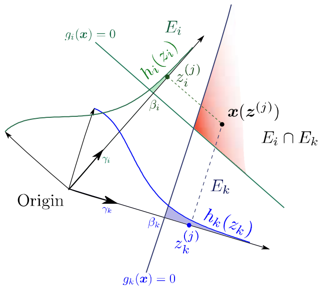

Multiple failure modes System reliability

Efficient Importance sampling

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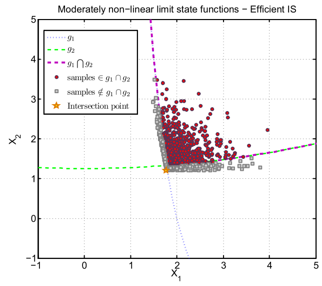
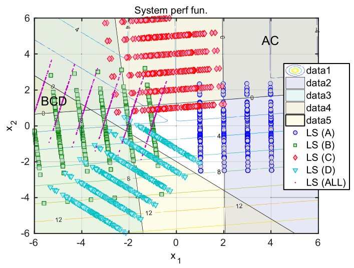


Simulation methods

Line sampling and Subset simulation

All the methods presented can be applied to estimate the failure probability of individual failure mode
Subset simulation should be able to identify different failure mode (in theory).
In practice there is no guarantee



Outline

Summary: computational tools

Analytical approaches:
- Limited to quasi-linear cases and low dimensions

Monte Carlo method
- Always applicable but requires large number of samples

Importance Sampling
- Requires prior-knowledge of important area

Line sampling
- Independent by the target probability level,
- Does not work for strong non linear performance function

Subset simulation
- Applicable for linear and non linear cases but difficult to tune

Summary

Which tools?

i High-dimensional:
Monte Carlo, Line sampling, Subset simulation

ii Multiple failure modes:
Monte Carlo, decompose failure modes → IS,LS

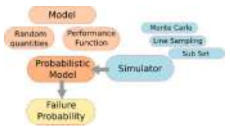
iii Small failure probabilities:
Line sampling (moderately non-linear), Subset simulation (otherwise)

OpenCossan

www.cossan.co.uk

Computational methods and examples part of OpenCossan

- Free and open source and **human readable** software
- Modular MATLAB® toolbox: easy to reuse components




```
%{
    This file is part of OpenCossan <https://cossan.co.uk>.
    Copyright (C) 2006-2018 COSSAN WORKING GROUP
%}
% Import package
import opencossan.reliability.*
% Define a Model
MyModel=opencossan.common.Model('Evaluator',MySolver,'Input',MyInputObject);
% Define the Performance function
MyPerFun=PerformanceFunction('Demand','maxDisplacement',...
                  'Capacity','displacementCapacity','OutputName','Vg');
% Define the Probabilistic model
MyProbMod=ProbabilisticModel('Model',MyModel,'PerformanceFunction',MyPerFun);
% Define the reliability solver
MySimulator=LineSampling('Lines',50);
% Perform reliability analysis
Pf=MyProbMod.computeFailureProbability(MySimulator);
```


Reliable engineering computing (REC2018)

Theme: **Computing with Confidence**
16-18 July 2018
www.rec2018.uk

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HYPER-SPHERICAL IMPORTANCE SAMPLING AND EXTRAPOLATION FOR HIGH-DIMENSIONAL RELIABILITY PROBLEMS

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In order to overcome challenges in low-probability, high-dimensional reliability problems (potentially with multiple failure domains), the speaker has been developing various reliability analysis methods recently. The presentation in this workshop will focus on two methods developed based on hyper-spherical description of high-dimensional reliability problems: (1) cross-entropy-based adaptive importance sampling using a von Misers-Fisher mixture model (Wang and Song, 2016); and (2) hyper-spherical extrapolation methods (Wang and Song, under review). The presentation will introduce the two methods in detail and present their performances in various numerical examples in order to identify merits and future research topics of the hyper-spherical approaches.

References:

[1] Wang, Z., and J. Song (2016). Cross-entropy-based adaptive importance sampling using von Mises–Fisher mixture for high dimensional reliability analysis. Structural Safety, 59:42-52.

[2] Wang, Z., and J. Song (under review). Hyper-spherical extrapolation method (HEM) for general high dimensional reliability problems. Structural Safety.

TNO innovation for life

TNO Workshop: Computational Challenges in the Reliability Assessment of Engineering Structures
Delft, The Netherlands

Hyper-spherical Importance Sampling and Extrapolation for High Dimensional Reliability Problems

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High dimensional Euclidean space

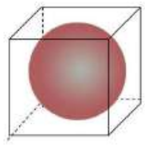
Volume Explosion

In n-dimensional space, consider a hypersphere inscribed in a hypercube

$$V_{hypersphere} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} R^n$$

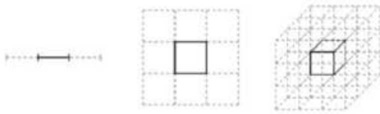
$$V_{hypercube} = (2R)^n$$

$$\frac{V_{hypersphere}}{V_{hypercube}} = \frac{\pi^{n/2}}{2^n \Gamma\left(\frac{n}{2}+1\right)} \to 0, n \to +\infty$$



Volume Concentration

Volume tends to distribute in the 'tails'



Betancourt (2017)

High dimensional probability space

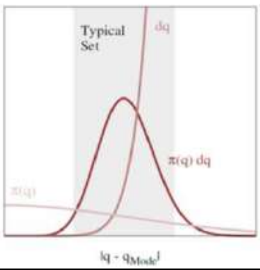
There may exist a typical set

In n-dimensional space, consider the probability

$$\Pr(\boldsymbol{q} \in \Omega) = \int_{q \in \Omega} \pi(\boldsymbol{q})\, d\boldsymbol{q}$$

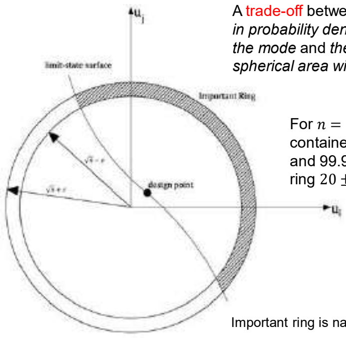
PDF $\pi(\boldsymbol{q})$ concentrates around its mode,
$d\boldsymbol{q}$ is much larger away from the mode

Betancourt (2017)



High dimensional standard normal space

The typical set is a hyper-ring



A trade-off between the exponentially decrease in probability densities with the distance from the mode and the exponentially increase in the spherical area with the distance from the mode

For $n = 400$, 95% probability is contained within the ring 20 ± 1, and 99.99% is contained within the ring 20 ± 2.

Important ring is named by Katafygiotis and Zuev (2008)

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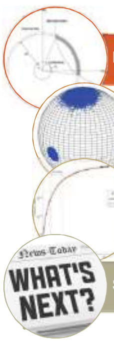

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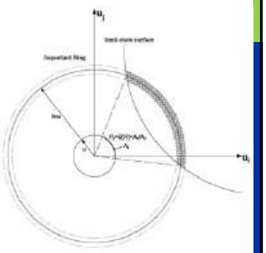
Hyper-spherical formulation

$$P_f = \int_0^\infty \theta(r) f_\chi(r) dr \cong \frac{1}{M}\sum_{i=1}^{M} \theta(r_i)$$

where $\theta(r) = A_f(r)/A_n$, $A_n = \frac{n\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$



- Valid for any dimensions
- Especially convenient for high dimensional problems

  r_i drawn from $f_\chi(r)$ is likely to have $r_i \in [\sqrt{n}-\varepsilon, \sqrt{n}+\varepsilon]$.

  Variation of $\theta(r_i)$ with r_i (drawn from $f_\chi(r)$) is expected to be small

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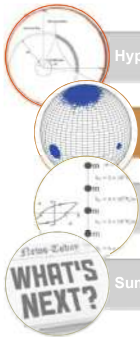


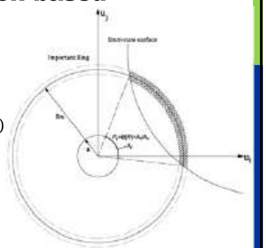
Hyper-spherical formulation based importance sampling

$$P_f = \int_0^\infty \theta(r) f_\chi(r) dr \cong \frac{1}{M}\sum_{i=1}^{M} \theta(r_i)$$

Construct an IS density to estimate $\theta(r_i)$

$$\theta(r_i) = \int \frac{I_{r_i}(r_i \bar{\mathbf{u}})}{A_n} d\bar{\mathbf{u}}$$

$$= \int \frac{I_{r_i}(r_i \bar{\mathbf{u}})}{A_n f_{IS}(\bar{\mathbf{u}})} f_{IS}(\bar{\mathbf{u}}) d\bar{\mathbf{u}} \cong \frac{1}{N}\sum_{j=1}^{N} \frac{I_{r_i}(r_i \bar{\mathbf{u}}_j)}{A_n f_{IS}(\bar{\mathbf{u}}_j)}$$



Finally, the IS formula is derived as

$$P_f \cong \frac{1}{N \cdot M}\sum_{i=1}^{M}\sum_{j=1}^{N} \frac{I_{r_i}(r_i \bar{\mathbf{u}}_j)}{A_n f_{IS}(\bar{\mathbf{u}}_j)}$$

where r_i drawn from $f_\chi(r)$, $\bar{\mathbf{u}}_j$ drawn from $f_{IS}(\bar{\mathbf{u}})$

Von Mises-Fisher Mixture as the IS density

Wang, Z., and Song J.(2016). Cross-entropy-based adaptive importance sampling using von Mises–Fisher mixture for high dimensional reliability analysis. *Structural Safety*. 59: 42-52.

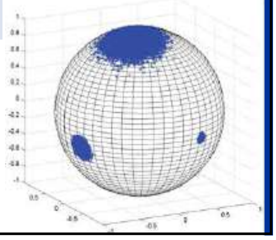
- **Sampling by "von Mises-Fisher Mixture" model**

$$f_{\mathrm{vMFM}}(\bar{\mathbf{u}}; \mathbf{v}) = \sum_{k=1}^{K} \alpha_k f_{\mathrm{vMF}}(\bar{\mathbf{u}}; \mathbf{v}_k)$$

where $\sum_{k=1}^{K} \alpha_k = 1$, $\alpha_k > 0$ for $\forall k$

$$f_{\mathrm{vMF}}(\bar{\mathbf{u}}) = c_d(\kappa) e^{\kappa \boldsymbol{\mu}^T \bar{\mathbf{u}}}$$

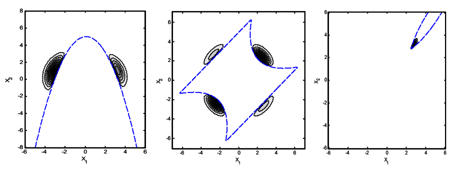
- κ: concentration parameter
- $\boldsymbol{\mu}$: mean direction
- α_k: weight for the k-th vMF



How can we find parameters of the vMFM model?

"Best" importance sampling density

$$p^*(\mathbf{x}) = \frac{|H(\mathbf{x})|}{\int |H(\mathbf{x})| d\mathbf{x}} = \frac{I(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})}{P_f}$$



- Can't use directly… if we already know P_f, we do not need MCS or IS.
- Still helpful for improving efficiency, if $h(\mathbf{x})$ is chosen in order to have **a shape similar to that of $I(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})$**

Adaptive importance sampling by minimizing cross entropy

Kullback-Leibler "Cross Entropy" (CE)

$$D(p^*, h) = \int p^*(\mathbf{x}) \ln p^*(\mathbf{x}) d\mathbf{x} - \int p^*(\mathbf{x}) \ln h(\mathbf{x}) d\mathbf{x}$$

- "Distance" between "best" IS density $p^*(\mathbf{x})$ and current one $h(\mathbf{x})$
- One can find a good $h(\mathbf{x})$ by minimizing Kullback-Leibler CE, i.e.

$$\arg\min_{\mathbf{v}} D(p^*, h(\mathbf{v})) = \arg\min_{\mathbf{v}} \left[\int p^*(\mathbf{x}) \ln p^*(\mathbf{x}) d\mathbf{x} - \int p^*(\mathbf{x}) \ln h(\mathbf{x}; \mathbf{v}) d\mathbf{x} \right]$$

$$= \arg\max_{\mathbf{v}} \int p^*(\mathbf{x}) \ln h(\mathbf{x}; \mathbf{v}) d\mathbf{x}$$

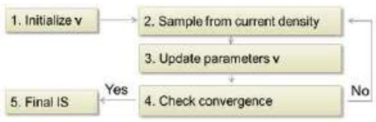
$$= \arg\max_{\mathbf{v}} \int I(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) \ln h(\mathbf{x}; \mathbf{v}) d\mathbf{x}$$

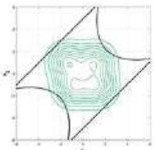
- Finds the optimal values of the distribution parameter(s) **v approximately by small-size pre-sampling**, then performs final importance sampling
- Rubinstein & Kroese (2004) used **uni-modal parametric distribution** for $h(\mathbf{x};\mathbf{v})$ and provided **updating rules** to find optimal **v** through sampling

CE-AIS with Gaussian Mixture (Kurtz & Song 2013)

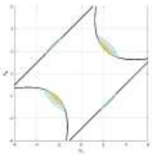
Kurtz, N., and Song J. (2013). Cross-entropy-based adaptive importance sampling using Gaussian mixture. *Structural Safety*. 42:35-44.

- **CE-AIS-GM Algorithm** $h(\mathbf{x};\mathbf{v}) = \sum_{k=1}^{K} \pi_k N(\mathbf{x} \mid \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$



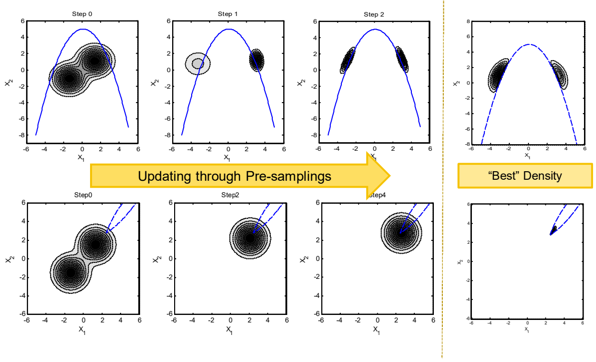


Near Optimal Density



Optimal Density

CE-AIS with Gaussian Mixture (Kurtz & Song 2013)



Parameter estimation for vMFM model

$$\alpha_k = \frac{\sum_{j=1}^{N} I_{r_i}(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j;\mathbf{w}) \Upsilon_j(z_k)}{\sum_{j=1}^{N} I_R(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j;\mathbf{w})}$$

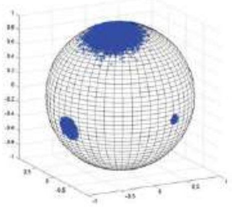
$$\boldsymbol{\mu}_k = \frac{\sum_{i=1}^{N} I_{r_i}(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j;\mathbf{w}) \Upsilon_j(z_k)\bar{\mathbf{u}}_j}{\left\|\sum_{i=1}^{N} I_{r_i}(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j;\mathbf{w}) \Upsilon_j(z_k)\bar{\mathbf{u}}_j\right\|}$$

$$\kappa_k \cong \frac{\xi n - \xi^3}{1-\xi^2}$$

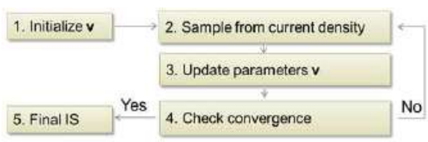
where

$$\xi = \frac{\left\|\sum_{i=1}^{N} I_{r_i}(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j;\mathbf{w}) \Upsilon_j(z_k)\bar{\mathbf{u}}_j\right\|}{\sum_{i=1}^{N} I_{r_i}(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j;\mathbf{w}) \Upsilon_j(z_k)}$$

$$\Upsilon_j(z_k) = \frac{\alpha_k f_{\mathrm{vMF}}(\bar{\mathbf{u}}_j;\mathbf{v}_k)}{\sum_{k=1}^{K} \alpha_k f_{\mathrm{vMF}}(\bar{\mathbf{u}}_j;\mathbf{v}_k)}$$



Procedures of Hyper-spherical importance sampling using vMFM



1. Pre-sampling to obtain near-optimal (i.e. minimum CE) vMFM sampling density using updating rules
2. Perform the final IS on hyper-spheres with radius drawn from the $f_\chi(r)$

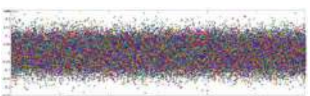
Example 1: Series system reliability in high-dimension

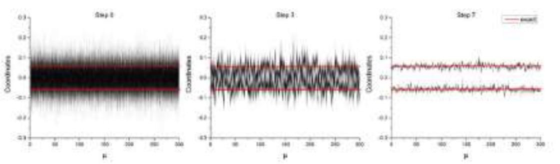
$$\mathrm{G}_1(\mathbf{u}) = \beta_1\sqrt{n} - \sum_{i=1}^{n}\mathbf{u}_i,\ \mathrm{G}_2(\mathbf{u}) = \beta_2\sqrt{n} + \sum_{i=1}^{n}\mathbf{u}_i$$

System failure domain: $\mathrm{G}_1(\mathbf{u}) \le \mathbf{0} \cup \mathrm{G}_2(\mathbf{u}) \le \mathbf{0}$

$\beta_1 = \beta_2 = 3.5,\ n = 300$

Updating of mean directions:





Example 2: Nonlinear random vibration analysis of MDOF system

- **Discrete representation of stochastic process representing ground acceleration (in frequency domain)**

$$\ddot{U}_g(t) = \sum_{j=1}^{n/2} \sigma_j [\mathrm{u}_j \cos(\omega_j t) + \hat{\mathrm{u}}_j \sin(\omega_j t)]$$

where

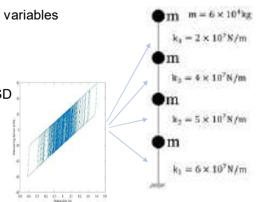
$\mathrm{u}_j, \hat{\mathrm{u}}_j$: independent standard normal random variables

ω_j: discretized frequency points

$\sigma_j = \sqrt{2S(\omega_j)\Delta\omega}$

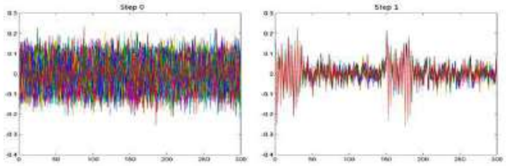
$S(\omega_j)$: two-sided power spectrum density/PSD

$\Delta\omega$: frequency step size

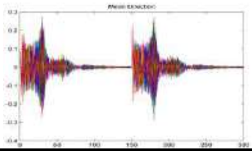


Example 2: Updating of vMFM

- **Instantaneous failure**



- **First-passage failure (series system)**



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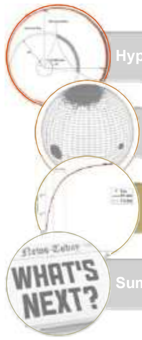


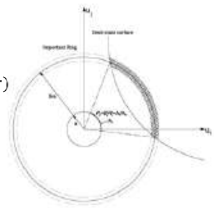
Hyper-spherical formulation based extrapolation

$$P_f = \int_0^{\infty} \theta(r) f_{\chi}(r) dr$$

Build an extrapolation method via writing $\theta(r)$ as $\theta(r) \cong \hat{\theta}(r, \mathbf{v})$

$$P_f = \int_0^{\infty} \theta(r) f_{\chi}(r) dr$$

$$\cong \int_{\sqrt{n}-\varepsilon}^{\sqrt{n}+\varepsilon} \hat{\theta}(r, \mathbf{v}) f_{\chi}(r) d\,r$$



Observe that $\theta(r)$ grows larger if r increases, given the safe domain is star-shaped with respect to the origin

Concept of the extrapolation:

- Find $\mathbf{v}$ of $\hat{\theta}(r, \mathbf{v})$ given $\theta(r)$ estimated from large radius r
- Estimate P_f using the hyper-spherical formulation

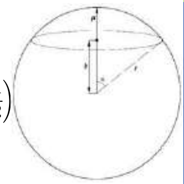
Model for failure ratio $\hat{\theta}(r, \mathbf{v})$

Wang, Z., and Song J. (2018). Hyper-spherical extrapolation method (HEM) for general high dimensional reliability problems. Structural Safety, 72: 65–73.

$$\theta_{cap}(r, \alpha) = \frac{A_{cap}(r, \alpha)}{A_n(r)} = \frac{1}{2} B_{sin^2\alpha}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

$B_{sin^2\alpha}(\cdot)$ is a regularized incomplete beta factor

$$\hat{\theta}(r, \alpha_k, K) = \sum_{k=1}^{K} \theta_{cap,k}(r, \alpha_k) = \frac{1}{2} \sum_{k=1}^{K} B_{sin^2\alpha_k}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$



Considering the dependence of α_k on r

$$\hat{\theta}(r, b_k, K) = \frac{1}{2} \sum_{k=1}^{K} B_{1-\left[\frac{b_k(r)}{r}\right]^2}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

Assume $b_k(r)$ does not change dramatically with r

- *Zeroth-order hyper-spherical extrapolation method* (ZO-HEM): $b_k(r) = b_k$
- *First-order hyper-spherical extrapolation method* (FO-HEM): $b_k(r) = a_k r + b_k$

Probability estimation using HEM

- **ZO-HEM:**

$$P_f \cong \sum_{k=1}^{K} \Phi(-b_k)$$

- **FO-HEM:**

$$P_f \cong \frac{1}{2} \int_{\sqrt{n}-\varepsilon}^{\sqrt{n}+\varepsilon} \sum_{k=1}^{K} B_{1-\left(a_k + \frac{b_k}{r}\right)^2}\left(\frac{n-1}{2}, \frac{1}{2}\right) f_{\chi}(r) d\,r$$

Procedures of HEM

- Select a sequence of m radii r_i, $i = 1, \ldots, m$, $r_i \in [r_{low}, r_{up}]$
- For each r_i, compute the failure ratio $\hat{\theta}(r_i)$
- Given $\hat{\theta}(r_i)$, compute optimal values of b_k and K in for ZO-HEM, or a_k, b_k and K for FO-HEM, so that the error function $\sum_{i=1}^{m} w_i \left[\log \hat{\theta}(r_i) - \log \theta(r_i)\right]^2$ is minimized, where w_i is a weight that puts more emphasis on more reliable data
- Compute the failure probability using CDF of standard normal distribution or numerical integration

Example 1: Series system reliability in high-dimension

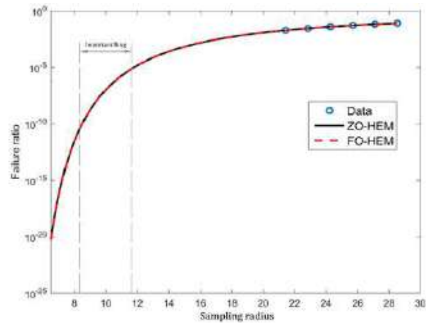
$$G_1(\mathbf{u}) = \beta_1\sqrt{n} - \sum_{i=1}^{n}\mathbf{u}_i, \; G_2(\mathbf{u}) = \beta_2\sqrt{n} + \sum_{i=1}^{n}\mathbf{u}_i$$

System failure domain: $G_1(\mathbf{u}) \le \mathbf{0} \cup G_2(\mathbf{u}) \le \mathbf{0}$

β_0	ZO-HEM			FO-HEM			Exact
	$\hat{\beta}$	c.o.v	Error (%)	$\hat{\beta}$	c.o.v	Error (%)	β
3.0	2.784	0.051	0.07	2.800	0.053	0.65	2.782
3.5	3.328	0.022	0.51	3.338	0.058	0.82	3.311
4.0	3.820	0.019	-0.33	3.846	0.043	0.33	3.833
4.5	4.366	0.009	0.36	4.381	0.025	0.71	4.350
5.0	4.906	0.052	0.86	4.894	0.051	0.59	4.865

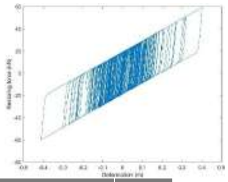
Example 1: Series system reliability in high-dimension

$\theta(r)$ versus r curves for $\beta_0 = 5.0$



Example 2: Nonlinear random vibration analysis of SDOF system

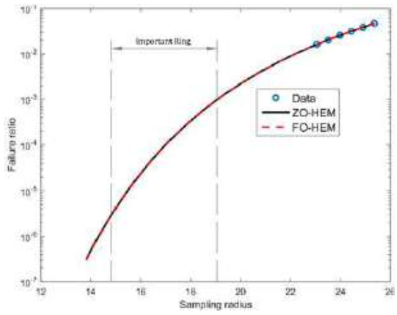
SDOF Bouc-Wen oscillator subjected to white noise



Threshold (m)	ZO-HEM			FO-HEM			Exact
	$\hat{\beta}$	c.o.v	Error (%)	$\hat{\beta}$	c.o.v	Error (%)	β
0.08	2.480	0.025	-2.95	2.518	0.043	-1.48	2.556
0.09	2.953	0.035	-2.72	2.971	0.048	-2.13	3.036
0.10	3.401	0.031	-3.92	3.475	0.037	-1.84	3.540

Example 2: Nonlinear random vibration analysis of SDOF system

$\theta(r)$ versus r curves for 0.10 (m) threshold



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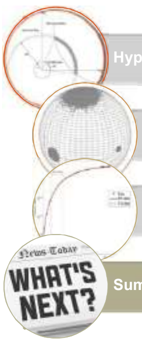
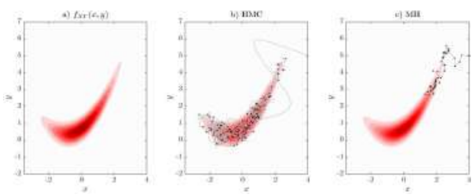


- Hyper-spherical formulation
- Hyper-spherical formulation based importance sampling
- Hyper-spherical formulation based extrapolation
- Summary and future research

Future research

- **[Possibilities]** Integration with Hamiltonian Monte Carlo based subset simulation

  Wang Z, Broccardo M, Song J. Hamiltonian Monte Carlo Methods for Subset Simulation in Reliability Analysis. arXiv:1706.01435



Summary

- **[Summary 1]** A hyper-spherical formulation to perform reliability analysis in high dimensional Gaussian space is proposed.
- **[Summary 2]** An importance sampling method using the hyper-spherical formulation in conjunction with von Mises-Fisher mixture distribution is proposed.
- **[Summary 3]** An extrapolation method using the the hyper-spherical formulation is proposed.

Wang, Z., and Song J.(2016). Cross-entropy-based adaptive importance sampling using von Mises–Fisher mixture for high dimensional reliability analysis. *Structural Safety*. 59: 42-52.

Wang, Z., and Song J. (2018). Hyper-spherical extrapolation method (HEM) for general high dimensional reliability problems. Structural Safety, 72: 65–73.

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MANY BETA POINTS TOO FAR: IS 42 REALLY THE ANSWER?

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Approach your problems from the right end
and begin with the answers. Then one day,
perhaps you will find the final question.
R. van Gulik, The Chinese Maze Murders

The classical problem of structural reliability is that for a limit state function (LSF) $g(\mathbf{x})$ in the n-dimensional Euclidean space and a probability distribution defined by a probability density function (PDF) $f(\mathbf{x})$ the probability of failure is defined as an integral:

$$P(F) = \int_{g(\mathbf{x})\leq 0} f(\mathbf{x})\, d\mathbf{x}$$

Most methods transform the problem from the original space to the standard normal space which yields:

$$\mathrm{P}(F) = (2\pi)^{-n/2} \int_{g(\mathbf{u})\leq 0} \exp\left(-\frac{|\mathbf{u}|^2}{2}\right) d\mathbf{u}.$$

Now several points will be discussed:

- Some philosophy. What is the problem seen in larger context? During the last fifty years the problem described in the last equation has changed, even if the formulation remained the same. Here gestalt switches occur not because we change our point of view, but because the structure we are studying changes. What was it and what is it now? Is the information we want to find numbers or structures? Plea for a structuralist view.

- Definition of the problem as a global minimization problem. Using the structure of the standard normal probability space one can define the problem as finding specific submanifolds on hyperspheres.

- Does a method which claims that the structure of the problem is irrelevant as subset sampling really work? This is a cautionary tale about a method without a clear mathematical concept.

- A tentative proposal for a solution. In the original FORM/SORM concept the design point is searched by solving the Lagrangian system:

$$\begin{aligned} \mathbf{u} + \lambda \nabla g(\mathbf{u}) &= \mathbf{0} \\ g(\mathbf{u}) &= 0 \end{aligned}$$

  Now, instead one searches the extrema of the LSF on a centered sphere with radius γ

$$\begin{aligned} \nabla g(\mathbf{u}) + \mu \mathbf{u} &= \mathbf{0} \\ |\mathbf{u}|^2 - \gamma^2 &= 0 \end{aligned}$$

  Going outside from a sphere where the minimum is larger than zero, one can reach by iteration a sphere where the minimum is equal to zero. For large dimensions then the probability mass of the set {g(u) ≤ 0} lies on a thin shell outside of this sphere.

Many beta points too far: is 42 really the answer?

Karl Breitung

ERA group, TU München
breitu@aol.com

A good decision is based on knowledge and not on numbers.
Plato

Thanks to Prof. S. Schäffler (UniBw Munich/Neubiberg) for explaining to the ignorant author some concepts of global optimization
(but he is not responsible for anything said here)

Terminus and Mike Box

The god of boundaries and limits
All should know their limits

Terminus and Mike Box

The god of boundaries and limits
All should know their limits

There is no strength in numbers,
have no such misconception.
(Uriah Heep, Lady in Black)

I. Structuralism

(What I think IMHO should be done)

Gestalt Switches

Thomas S. Kuhn

Kuhn argued in *The structure of scientific revolutions* Kuhn (1996) that these are caused by *gestalt switches*. One looks at the known fact or structure from different angle or perspective and suddenly one sees something different. But also in the time between revolutions science progresses by many small *gestalt switches* (see Kuhn (1996), p. 181 and Kuhn (1970), p. 249, note 3). Also in structural reliability there was a sequence of such switches.

Structuralism

Jean Piaget

Structuralism is a scientific methodology emphasizing the relations between the elements of the subject as main topic of the study, for a description see Piaget (1971). After Rickart (1995) "structuralism" can be defined as a *method of analyzing a body of information with respect to its inherent structure*.
Mathematical structuralists think that mathematics is fundamentally concerned with structures, or with the relations mathematical objects bear to each other in virtue of belonging to some structure.

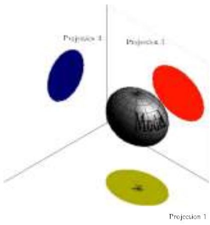
Charles E. Rickart

A Gestalt Switch towards Structuralism

Structural reliability should make a gestalt switch towards a structuralist view of reliability problems. This becomes more and more necessary, since the problem structures are getting more complex.
Try to identify the relevant substructure as primary target, failure probabilities then as secondary target.

The changing shapes

Von der Vergangenheit trennt uns nicht ein Abgrund, sondern die veränderten Verhältnisse. (A.Kluge)



(a) The reliability problem at the beginning

The changing shapes

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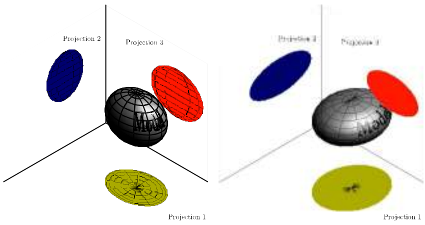
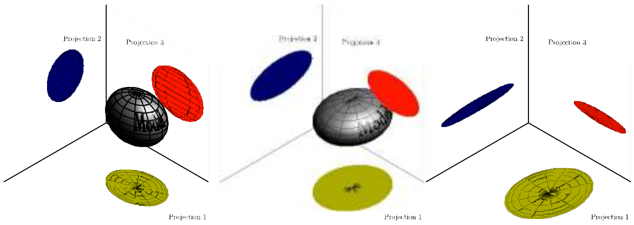


(a) The reliability problem at the beginning (b) The reliability problem evolving

The changing shapes

Von der Vergangenheit trennt uns nicht ein Abgrund, sondern die veränderten Verhältnisse. (A.Kluge)

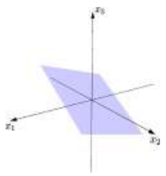


(a) The reliability problem at the beginning (b) The reliability problem evolving (c) The reliability problem now

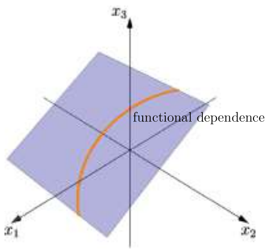
Figure: The varying shapes of the reliability problem

Development of structural model

Underlying Hilbert space



Dimension reduction



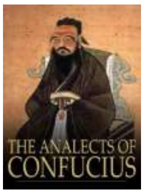
Building a functional model

II. Subset Simulation

(Which IMHO is wrong)

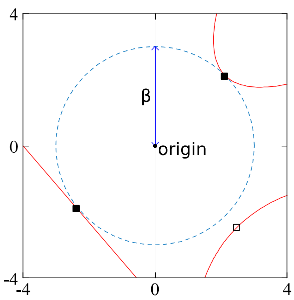
Confucius on Names

One day, a disciple asked Confucius: "If a king were to entrust you with a territory which you could govern according to your ideas, what would you do first?"
Confucius replied: "My first task would certainly be to rectify the names."
The puzzled disciple asked: "Rectify the names? Is this a joke?"
Confucius replied: "If the names are not correct, if they do not match realities, language has no object. If language is without an object, action becomes impossible..."
(The Analects of Confucius, Book 13, Verse 3)



The Basic Problem

In standard normal space with pdf $f(\mathbf{u}) = (2\pi)^{-n/2}\exp(-|\mathbf{u}|^2/2)$ approximate $P(F) = \mathrm{P}(\{g(\mathbf{u}) < 0\})$. This is the *REAL THING*, nothing else, and also SuS is an approach to solve this.



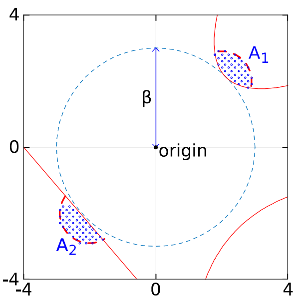
In the standard normal space the design points (filled black squares) have to be found. Then with FORM/SORM asymptotic approximations are derived:

$$\mathrm{P}(F) \sim \Phi(-\beta)\cdot C,\ \beta \to \infty$$

F(S)ORM First (Second) Order Reliability Methods referring to the order of the Taylor expansion.

The Basic Problem in SuS Formulation

In standard normal space with pdf $f(\mathbf{u}) = (2\pi)^{-n/2}\exp(-|\mathbf{u}|^2/2)$:



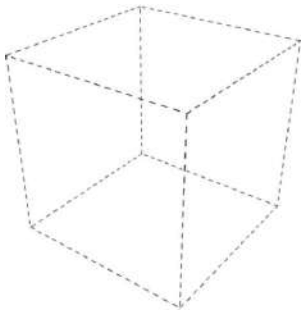
Doing asymptotic analysis without calculus. In the standard normal space the design areas A_1 and A_2 (neighborhoods of the design points) have to be found and their probability content estimated for an asymptotic approximation.

$$\mathrm{P}(F) \sim \mathrm{P}(A_1) + \mathrm{P}(A_2),\ \beta \to \infty$$

This is a result derived by M. Hohenbichler (see Breitung (1994), p. 53).

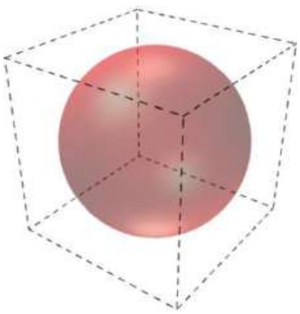
Mathematical and engineering logic I

The cube denotes a set of problems. Assume a mathematician finds a solution idea. He will derive a theorem valid in the red sphere.



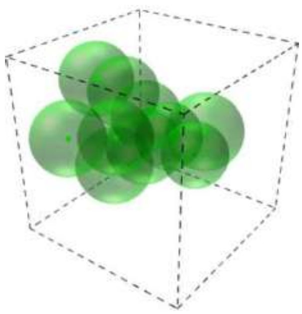
Mathematical solution I

The cube denotes a set of problems. Assume a mathematician finds a solution. He will derive a theorem valid in the red sphere.
An engineer will check his heuristics



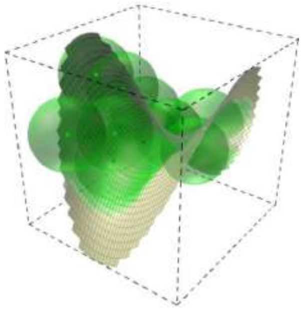
Engineering solution I

An engineer will check his solution idea by calculating a number of examples (green dots). So he will get an idea that the method works for similar cases (green spheres).



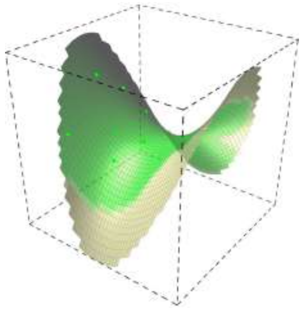
Hidden assumption I

But since in the calculation of these examples it is not clearly specified what properties these examples have, it might happen that there is a hidden assumption common to all examples (grey surface).



¡restricted validity I

So in fact taking into account this hidden assumption, the method is valid only for the cases where this assumption is fulfilled.
Green surface part.



Credo of Subset Simulation (SuS)

Zuev et al. (2012):
Subset Simulation provides an efficient stochastic simulation algorithm for computing failure probabilities for general reliability problems without using any specific information about the dynamic system other than an input-output model. This independence of a systems inherent properties makes Subset Simulation potentially useful for applications in different areas of science and engineering where the notion of "failure" has its own specific meaning,...

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Monahan (2011) p. 394:
...For MCMC, an extremely naive user can generate a lot of output without even understanding the problem. The lack of discipline of learning about the problem that other methods require can lead to unfounded optimism and confidence in the results.

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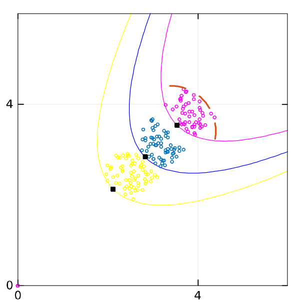
Monahan (2011) p. 394:
...For MCMC, an extremely naive user can generate a lot of output without even understanding the problem. The lack of discipline of learning about the problem that other methods require can lead to unfounded optimism and confidence in the results.

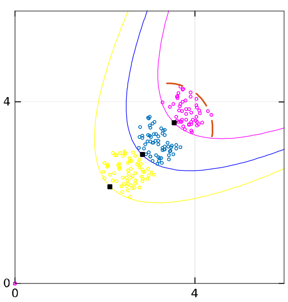
Promise of SuS: You can do everything without understanding anything!

The Standard SuS Example



From a larger value $c_1 > 0$ the failure regions $F_j = \{g(\mathbf{u}) < c_j\}$ with $c_1 > c_2 > ..c_n = 0$ are made successively smaller until the original failure domain $\{g(\mathbf{u}) < 0\}$ is reached. Here also the design points for the domains F_j are shown. Using Hohenbichler's lemma now estimate the probability from the points in magenta.

Iteration: Design Points and Regions



In the SS approach the relevant areas of F_n are found near the last region in F_{n-1} [a]. In SORM this corresponds to searching the next design point for F_j in the neighborhood of the last for F_{j-1}. *Sounds reasonable?*
A really grave problem in mathematics is that not everything which sounds reasonable is correct.

[a] *"Given that we have found a failure point $\theta \in F_{n-1}$, it is reasonable to expect that more failure points are located nearby"*

This does not work as advertised!

Some Warnings Ignored

Rüdiger Rackwitz

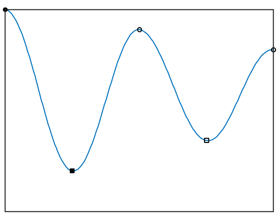
As Rackwitz (2001) said, an important step in the development of methods is to show where they do not work, i.e. to find the limits of the applicability of the concept and to construct counterexamples.

And Hooker (1995) said that the most important point is to understand an algorithm not to make it efficient. `http://repository.cmu.edu/tepper/197/`

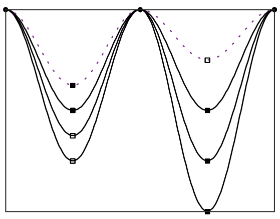
John N. Hooker

Sequential determination of global extrema

Global and local extrema of functions: minima are shown by squares, maxima by circles, filled symbols are global extrema



(a) Local and global extrema of a function



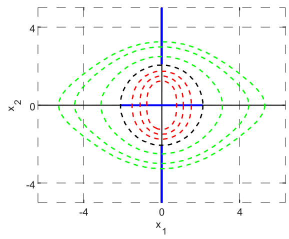
(b) The global minima of a function depending on a parameter

A Simple Example with Smooth Functions

The position of global minima under constraints. Given a LSF:

$$g(u_1,u_2) = \beta - u_1^2 - \frac{|\mathbf{u}|^2}{b^2}u_2^2 = \beta - u_1^2 - \frac{(u_1^2+u_2^2)u_2^2}{b^2} = \beta - u_1^2 - \frac{u_2^1+u_2^4}{b^2}$$

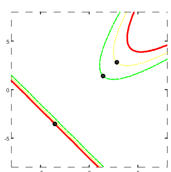
The points with global minimum distance to the origin under $g(u_1,u_2)=c$ lie always on the axes (on the blue line segments).



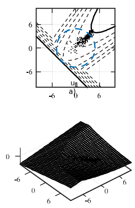
The minimum distance points jump when the black circle is reached.
This is a normal behavior in global optimization!

Series System with SuS

Find the point with minimal distance to the origin — design point — on the domain bounded by the thick red curve $\{g(\mathbf{u})=0\}$.



(a) The contours for g



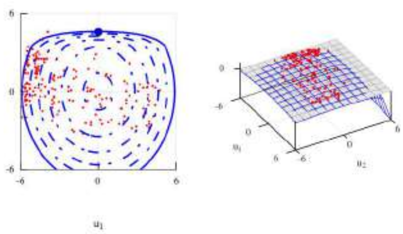
(b) SuS algorithm for g

$$g_1(u_1,u_2) = 0.6 + \frac{(u_1-u_2)^2}{40} - \frac{u_1+u_2}{10\sqrt{2}}, \qquad g_2(u_1,u_2) = 5 + \frac{u_1+u_2}{\sqrt{2}}$$

$$g(u_1,u_2) = \min(g_1,g_2)$$

Remember Wild Bill Hickok. You also have to look behind you!

Extrapolation with SuS



LSF: $g(x_1,x_2) = 0.1\cdot(52 - 1.5\cdot x_1^2 - x_2^2)$

$$F(x_1) = \Phi(x_1),\; F_2(x_2) = \begin{cases}\Phi(x_2) & , x_2 \le 3.5\\ 1-x_2^c & , x_2 > 3.5\end{cases}$$

Global minimization and SuS

It is not possible to find the design point (global minimum point on $g(\mathbf{u}) = 0$) by a sequential method for $c_1 > c_2 > \dots c_n = 0$

$$|\mathbf{u}^j| = \min_{g(\mathbf{u}) \le c_j} |\mathbf{u}|, \quad \mathbf{u}^j \to \mathbf{u}^{j+1} \to \mathbf{u}^n = \min_{g(\mathbf{u}) \le 0} |\mathbf{u}|$$

This works in examples with a *Simple Simon* geometry, but not in general.
If someone says, SuS is not an attempt to global minimization, **what is it then**?
And if someone says, SuS does not work for such simple examples, remember: *Hic Rhodus, hic salta!*
The main problem in global optimization is to avoid local extrema and to get out of them if stuck there. Unfortunately this is complicated, it is not enough to run some MCMC's and wait.

III. Onion Concept

(Which IMHO might help a little)

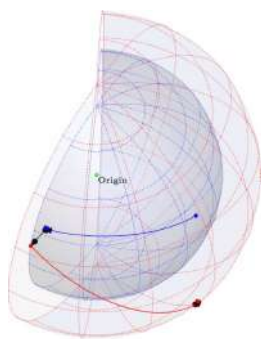
In global minimization for structural reliability one has to find the global minimum point $\mathbf{u}^*$:

$$|\mathbf{u}^*| = \min_{g(\mathbf{u}) \le 0} |\mathbf{u}|$$

Define the spheres $S(y) = \{\mathbf{u}; |\mathbf{u}| = y\}$, this can be done finding the beta sphere defined by

$$\beta = \min_{y>0} \{S(y); \min_{\mathbf{u} \in S} g(\mathbf{u}) \le 0\}$$

The Onion Concept



In the original FORM/SORM concept the design point is searched by solving the Lagrangian system:

$$\begin{aligned} \mathbf{u} + \lambda \nabla g(\mathbf{u}) &= \mathbf{0} \\ g(\mathbf{u}) &= 0 \end{aligned}$$

Now, instead one searches the extrema of the LSF on a centered sphere with radius γ in an iterative way

$$\begin{aligned} \nabla g(\mathbf{u}) + \mu \mathbf{u} &= \mathbf{0} \\ |\mathbf{u}|^2 - \gamma^2 &= 0 \end{aligned}$$

Onion Method Example

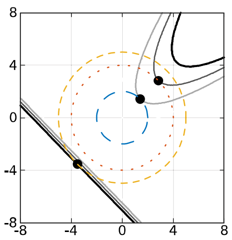


Figure: The contours for g

Table: Iteration steps

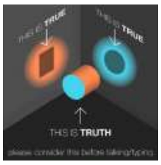
Step	Iteration Point
1	$(1, 2)$
2	$(-1.58, -1.58)$
3	$(-4.58, -4.58)$
4	$(-3.10, -3.10)$
5	$(-3.71, -3.71)$
6	$(-3.46, -3.46)$
7	$(-3.57, -3.57)$
8	$(-3.52, -3.52)$
9	$(-3.54, -3.54)$
10	$(-3.53, -3.53)$

IV. Philosophy of science

Against Method

Paul Feyerabend

This is not an appeal to go forward in a specific direction but to see things from a broader perspective and to try out various methods and concepts. Since science is — as Feyerabend (1993) says — in principle an anarchistic enterprise. And to give a further quote from him, all methodologies have their limits even the most obvious ones. So there is plenty of room for new research.



Instead of Conclusions an Advice from Star Trek

Episode *Phage* from Voyager

Kes: How does a real doctor learn to deal with patients' emotional problems, anyway?
The Doctor: They learn from experience.
Kes: Aren't you capable of learning?
The Doctor: I have the capacity to accumulate and process data, yes.
Kes: **Then I guess you'll just have to learn - like the rest of us.**

Thank you for your attention

Some manuscripts: Researchgate, arxiv, osf

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PART 2: PRACTICE

RESPONSE SURFACE METHODS AND RANDOM FIELDS COUPLED TO NONLINEAR FINITE ELEMENT ANALYSIS IN DIANA 10.2.

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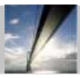
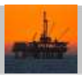
Diana

RESPONSE SURFACE METHODS and RANDOM FIELDS coupled to NONLINEAR FINITE ELEMENT ANALYSIS using DIANA





Civil Engineering
Geotechnical Engineering
Petroleum Engineering

DIANA FEA

2018, January 24

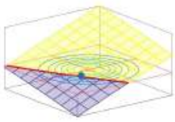
Overview

I. Response Surface Method coupled to FEA (Probab)
  1. Reliability analysis with DIANA
  2. Case study: Probabilistic analysis of reinforced concrete beam
  3. Case study: Reliability assessment of gravity dam block
  4. Discussion

II. Random Field coupled to FEA
  1. Random fields with DIANA
  2. Case study D: Concrete floor under restrained shrinkage
  3. Discussion

III. Challenges

I. Response Surface Methods coupled to FEA

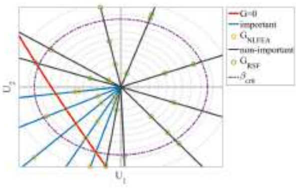


I. 1. Reliability analysis with DIANA

Directional Adaptive Response Surface Method (DARS)

- limit state function evaluations (LSFE)
  - NLFEA : $\beta_i \leq \beta_{crit}$
  - RSF : $\beta_i > \beta_{crit}$
- directional sampling in $\underline{U}$-space
  - $P_f = \frac{1}{N}\sum_{i=1}^{N}(1-\chi_n^2(\beta_i^2))$
  - $COV_{P_f} = \frac{1}{N(N-1)}\sum_{i=1}^{N}(P_i - P_f)$
- response surface function (RSF)

$$G^*(U_n) = \alpha + \sum_{i=1}^{n} b_i U_i + \sum_{i=1}^{n}\sum_{j=1}^{i} c_{ij} U_i U_j$$

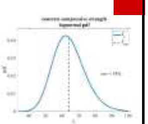


Based on the PhD Thesis of P. Waarts

I. Additional Case study

Concrete block(s) under monotonic loading
- performance
- analytical formulas / NLFEA convergence
- material uncertainty
- limit state surface

$$G\{f_c\} = F_{\max}\{f_c\} - F_{exam}$$

COV_{P_f}	sampled directions	no. of NLFEA	$G_{RSF} = 0$	design point	β	$\beta_{analytical}$
45%	24	34	7	3.88	**3.76**	**3.84**

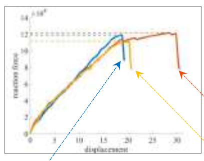
- model uncertainty, $\boldsymbol{\theta}$: $G\{f_c, \boldsymbol{\theta}\} = \boldsymbol{\theta} \cdot F_{\max}\{f_c\} - F_{exam}$

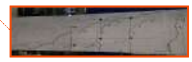
I. 2. Case study

Experimental results

- similar experiments
- different failure mode

⇨ material uncertainty





beam A – flexural failure

beam B – shear failure

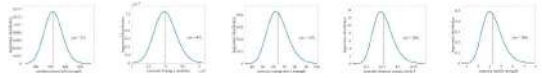
beam C – shear failure

Based on MSc Thesis by P. Evangeliou

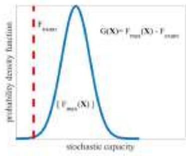
Based on MSc Thesis by P. Evangeliou

I. 2. Case study

- X-space



- limit state function
- tight tolerance
- ≈ 7800 dof's



Based on MSc Thesis by P. Evangeliou

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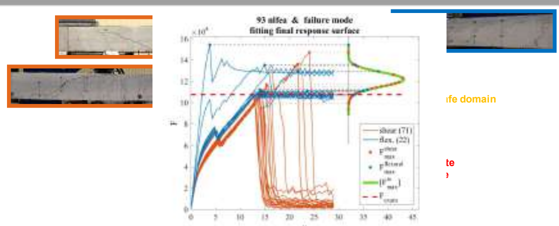
I. 2. Case study



$\beta = 2.22$, $LSFE_{NLFEA} = 200$, $LSFE_{RFS} = 1200$

Based on MSc Thesis by P. Evangeliou

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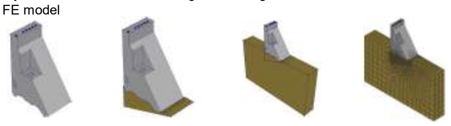
I. 3. Case study

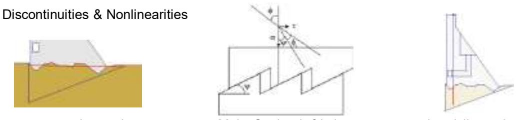
Reliability assessment of dam block against sliding failure — ICOLD 2017

- 3D FE model



≈ 143000 dof's

- Discontinuities & Nonlinearities

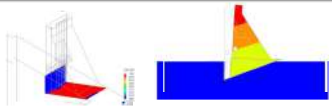


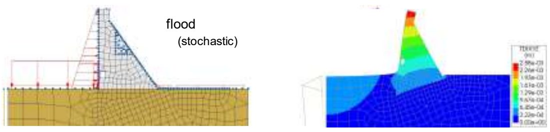
ψ - macroscopic roughness — Mohr-Coulomb friction — bondslip anchors

I. 3. Case study

Advanced FE modeling

- Phased analysis
- Staggered analysis
- Mohr-Coulomb zero-tension interfaces

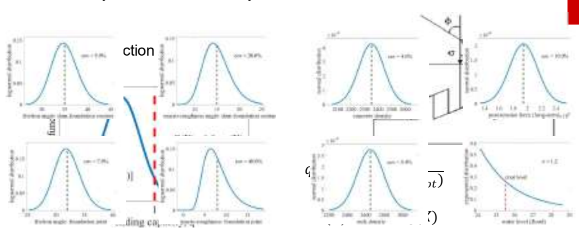




ICOLD Benchmark 2017

I. 3. Case study

- uncertainty: material & boundary



ICOLD Benchmark 2017

I. 3. Case study

Reliability assessment

- component
- system

ICOLD Benchmark 2017

I. 5. Discussion

Coupling DARS to NLFEA

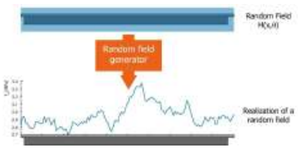
Robustness
- quality of RSF
- line search
- numerical stability
- model uncertainty

$$G\{\boldsymbol{X},\boldsymbol{\theta}\} = \boldsymbol{\theta} \cdot \mathrm{R}\{\boldsymbol{X}\} - \mathrm{S}$$

Efficiency
- parallel processing
- advanced sampling

Extend benchmarks

II. Random Fields coupled to FEA



II. 1. Random Fields with DIANA

JCSS material model
- integration points

Correlation function
- exponential
- squared exponential
- threshold value

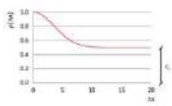
Distribution
- normal
- log-normal

Random Field generators
- Covariance Matrix Decomposition (CMD)
- Fast Fourier Transform (FFT)
- Local Average Subdivision (LAS)

$$\rho(\Delta x) = e^{-\frac{\Delta x}{L_c}}$$

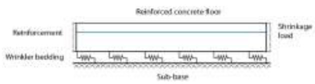
$$\rho(\Delta x) = e^{-\frac{\Delta x^2}{L_c^2}}$$

$$\rho(\Delta x) = c_1 + (1 - c_1) \cdot e^{-\frac{\Delta x^2}{L_c^2}}$$



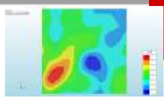
II. 2. Case study D

Mechanical scheme:



Random Field:
- JCSS: f_c (ρ = 1.0)
- log-normal
- FFT
- SqExp

compressive strength



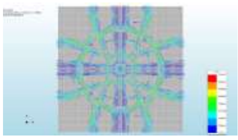
tensile strength

Young's modulus

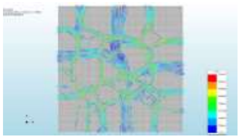
Based on MSc Thesis by R.v.d.Have

II. 2. Case study D

- Crack growth – no RF



- Crack growth – with RF



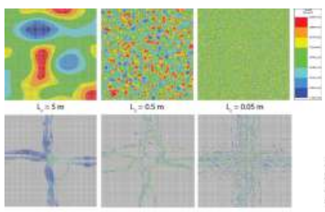
Based on MSc Thesis by R.v.d.Have

II. 2. Case study D

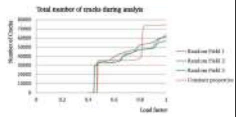
RF 1 RF 2 RF 3

tensile strength

crack strains



Number of cracks



Based on MSc Thesis by R.v.d.Have

II. 3. Discussion

RF coupling to NLFEA

- crack initialization at weakest point / asymmetric crack pattern
- numerical stability (?)
  - gradual development of cracking: convergence
  - cracking localization (ρ, COV)

RF parameters

III. Challenges

- Coupling RF to DARS to NLFEA
- Calibrate current safety formats
- Engineering practice

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DIANA FEA

RELIABILITY ANALYSIS OF REINFORCED CONCRETE STRUCTURES: ACCOMPLISHMENTS AND ASPIRATIONS.

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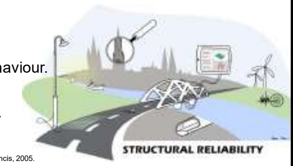
RELIABILITY ANALYSIS OF REINFORCED CONCRETE STRUCTURES: ACCOMPLISHMENTS AND ASPIRATIONS

Arthur Slobbe and Arpad Rozsas

TNO workshop: Computational Challenges in the Reliability Assessment of Engineering Structures
2018, Delft

MOTIVATION

- Built infrastructure: >50% of national wealth[1].
- Ageing infrastructure[2].
- Uncertainty of parameters, models[2].
- Aim:
  - Better understanding of structural behaviour.
  - Uncertainty reduction.
  - More economical asset management.



[1] Sarja A. (2005). *Integrated Life Cycle Design of Structures*. e-Library: Taylor & Francis, 2005.
[2] Frangopol, D. M. (2011). Life-cycle performance, management, and optimisation of structural systems under uncertainty: accomplishments and challenges. *Structure and Infrastructure Engineering*, 7(6), 389-413.

PROBLEM STATEMENT

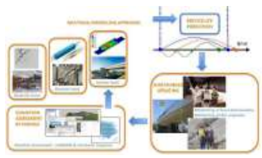
- Current practice and methods:
  - Almost exclusively for new structures.
  - How to deal with deterioration?
  - What to do with non-complying structures? (reserves?)
  - How to use NLFEA for verification?
- Reliability assessment can help, but:
  - NLFEA based limit state functions are **computationally expensive**.
  - Model uncertainty?
  - Random fields?
  - How to translate to methods usable in practice?

Probabilistic model

Physical (mechanical model)	No explicit prob. model	Some prob. models	All relevant variables are assigned a prob. model
Non-linear FEA	Glob. safety factor	E-COV; Novel method	Reli. analysis (level II/III)
Linear FEA	State of the current practice		
"Hand" calculation	State of the current practice		

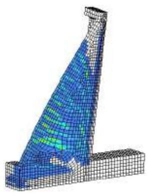
RELATED TNO PROJECTS

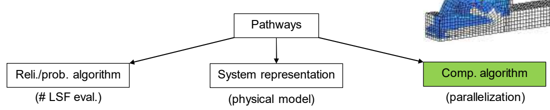
- Reinforced concrete bridges (deterioration).
- Hydraulic structures (soil-structure interaction + deterioration).
  - Advanced NLFEA
  - Probabilistic assessment:
    - Reliability assessment.
    - Structural health monitoring.
    - Probabilistic damage identification.
  - Probabilistic modelling of corrosion.



COMMON UNDERLYING CHALLENGES

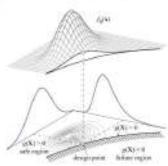
- Computational challenge: attain reasonable running times.
  - Multiple failure modes
  - Random fields
  - Probabilistic updating for existing structures.
  - Code calibration (optimization).





TOOLS – RELIABILITY ANALYSIS

- In-house:
  - Prob2B (+ Python + Openturns)
  - FERUM (Matlab) – extended:
    - Adaptive direction sampling[1]:
      - multiple of response surface types (polynomial, kriging, goodness-of-fit)
  - Coupling with FEM, e.g. Diana, OpenSees.



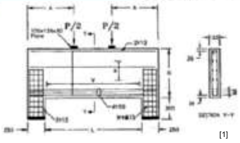
[1] Grooteman, F. (2011). An adaptive directional importance sampling method for structural reliability. *Probabilistic Engineering Mechanics*, 26(2), 134-141.

RC STRUCTURES
DEEP BEAM

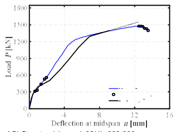
- High-strength RC deep beam.
- Bending failure & diagonal crack at support
- NLFEA in Diana
  - Plane stress elements.
  - Concrete: cracking, crushing.
  - Rebar: hardening and rupture (perfect bond).

- Limit state function (three random variables):

$$g = R(f_{cc}, f_{sy}) - S$$



[1]



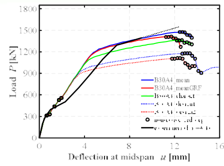
[1] Foster, S.J., Gilbert, R.I. (1998) Experimental Studies on High-Strength Concrete Deep Beams. ACI Structural Journal. 95(4). 382-390.

RC STRUCTURES
DEEP BEAM

Variable	*Distribution*	*Mean*	*Coefficient of variation*
Concrete compressive strength (f_{cc})	Lognormal	88 MPa	0.06
Rebar yield strength (f_{sy})	Lognormal	440 MPa	0.045
External load (S)	Gumbel	600 KN	0.20

Method	*# LSF evaluations*	*Running time*	*β*	*$α_i$*	*R* [kN]*
DARS[1]	66	2.5 hours*	3.87	{0.05, 0.15, -0.99}	1446

*Diana 10.1 and a Kepler computer was used for the analysis: 16GB Intel(R) Xeon(R) CPU E5-2620 v3 @2.40GHz; 2.40GHz; Windows 10 Enterprise 64-bit SP1.



RC STRUCTURES
DEEP BEAM

- Added value of more advanced modelling:

Probabilistic model

Physical (mechanical model)	No explicit prob. model	Some prob. models	All relevant variables are assigned a prob. model
Non-linear FEA	Glob. safety factor	E-COV; Novel method	Reli. analysis (level II/III)
Linear FEA	State of the current practice		
"Hand" calculation	State of the current practice		

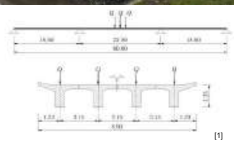
	analytical modelling	*nonlinear finite element modelling*			
		semi-probabilistic methods			*full-probabilistic method*
	EC	*PF*	*GRF*	*E-COV*	*DARS*
R_d [kN]	771	1110	1110	1196	-
S_d [kN]	1367	1367	1367	1367	equivalent prob. model
S_d / R_d [-]	1.77	1.23	1.23	1.14	-
$β$ [-]	-	-	-	-	3.87
$β/β_t$ [-]	-	-	-	-	1.02

RC STRUCTURES
CONTINUOUS BEAM

- Three-span beam.
- Bending failure (mechanism).
- NLFEA in Diana:
  - Beam-column elements.
  - Concrete: cracking, crushing.
  - Rebar: hardening and rupture (perfect bond).

- Limit state function (five random variables):

$$g = R(f_{cc}, f_{sy}, f_{su}, \varepsilon_{su}) - Q$$



[1]

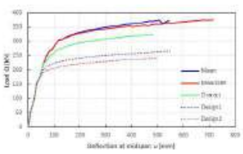
[1] Jacinto, L., Neves, L.A.C., Santos, L. (2015). Bayesian assessment of an existing bridge: a case study. *Structure and Infrastructure Engineering*, 12(1), 1-17.

RC STRUCTURES
CONTINUOUS BEAM

Variable	*Distribution*	*Mean*	*Coefficient of variation*
Concrete compressive strength (f_{cc})	Lognormal	51.2 MPa	0.07
Rebar yield strength (f_{sy})	Lognormal	440 MPa	0.065
Rebar ultimate strength (f_{su})	Lognormal	550 MPa	0.07
Rebar ultimate strain ($ε_{su}$)	Lognormal	0.08	0.09
External load (Q)	Gumbel	$1.5 \cdot 10^2$ kN	0.20

Method	*# of iteration/ simulation steps*	*# of limit state function evaluations*	*Reliability index, β*	*Total comp. time [hr]†*
DARS	-	113	3.80	12

† Diana 10.1 and a Kepler computer was used for the analysis: 16GB Intel(R) Xeon(R) CPU E5-2620 v3 @2.40GHz; 2.40GHz; Windows 10 Enterprise 64-bit SP1.



RC STRUCTURES
CONTINUOUS BEAM

- Added value of more advanced modelling:

	analytical modelling	*nonlinear finite element modelling*			
		semi-probabilistic methods			*full-probabilistic method*
	EC	*PF*	*GRF*	*E-COV*	*DARS*
R_d [kN]	293 (or 211)	242	295	271	-
S_d [kN]	342	342	342	342	equivalent prob. model
S_d / R_d [-]	1.62	1.41	1.16	1.26	-
$β$ [-]	-	-	-	-	3.80
$β/β_t$ [-]	-	-	-	-	1.00

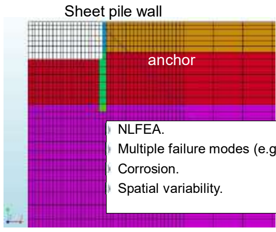
RC STRUCTURES

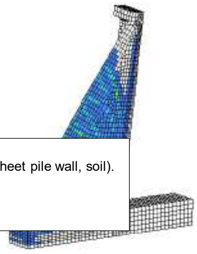
CONTINUOUS BEAM

- Extension (ongoing):
  - Random field representation of corrosion.
  - Pitting corrosion of rebars.
  - NLFEA also in OpenSees to reduce the calculation time.
- Promising results/directions from the literature:
  - GPU parallelization: 10x speed-up[1].
  - Cloud computing[2].

[1] Tian Y., Xie L., Xu Z., and Lu X.. (2015). GPU-Powered High-Performance Computing for the Analysis of Large-Scale Structures Based on OpenSees. 2015 *International Workshop on Computing in Civil Engineering*.
[2] Ventura C.E. and Bebamzadeh A. (2013). Los Angles Tall Buildings Structural Design Council. *Annual meeting*. Presentation slides.

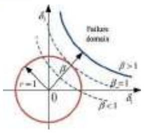
HYDRAULIC STRUCTURES





- NLFEA.
- Multiple failure modes (e.g. anchor, sheet pile wall, soil).
- Corrosion.
- Spatial variability.

CHALLENGES – QUESTIONS



- Computationally demanding physical models → reduce the computational time:
  - Parallelization.
  - "Smart" algorithms.
- Solving reliability problems with random fields (1D-2D-(3D)), (large number of correlated variables).
- Multiple failure modes.
- Reliability based calibration of methods for practice, e.g.:
  - NLFEA-based verification and/or design.
- We are looking, open for:
  - Join our efforts on implementing and testing algorithms/methods.
  - Benchmark problems (the showcased examples).
  - Joint effort to compile an open document with reliability methods (pros, cons).

MODEL UNCERTAINTY OF RESISTANCE MODELS OF RC STRUCTURES BASED ON NUMERICAL SIMULATIONS.

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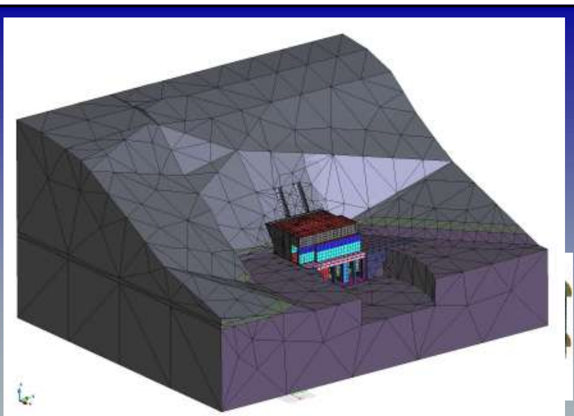
Červenka Consulting

UNCERTAINTY OF NUMERICAL MODELS OF RC STRUCTURES

Vladimir Cervenka
Cervenka Consulting, Prague

CONTENTS



FEM Nonlinear analysis of ULS → Global safety format

$F_d < R_d$ $\quad R_d = \frac{R_m}{\gamma_M \gamma_{Rd}}$

Gglobal safety factor for material uncertainty γ_M

Safety factor for model uncertainty γ_{Rd}

EN 1992 - 2	Model Code 2010
$\gamma_{Rd} = 1.06$	$\gamma_{Rd} = 1.0 \div 1.1$

Failure mode
Constitutive formulations

Model uncertainty as random variable
Method of assessment based on validation by experiments

$$\theta = \frac{R_{test}}{R_{simulation}}$$

Data base θ_i , i – no. of samples

Safety factor of model uncertainty

Large data set – log-normal PDF $\quad \gamma_{Rd} = \frac{\exp(\alpha_R \beta \times V_\theta)}{\mu_\theta}$

Limited data set – Student's PDF $\quad \gamma_{Rd} = \frac{1}{\mu_\theta \exp(t_{p=0.112}(n-1) \times V_\theta)}$

Case study ULS 33 cases

Punching shear tests 15
Guandalini, S. And Muttoni, A., EPFL, Lausanne
Hallgren M., KTH Stockholm

Shear strength of large beams 7
Collins M.P., Toronto

Bending strength of beams 11
Debernardi P.G., Torino

Range of parameters:

Modes of failure:
Brittle (concrete), ductile (steel)
With and without shear reinforcement

Concrete:
NSC, HSC

Size range:
0.1 to 4 m scale 1:40

Numerical simulation by ATENA

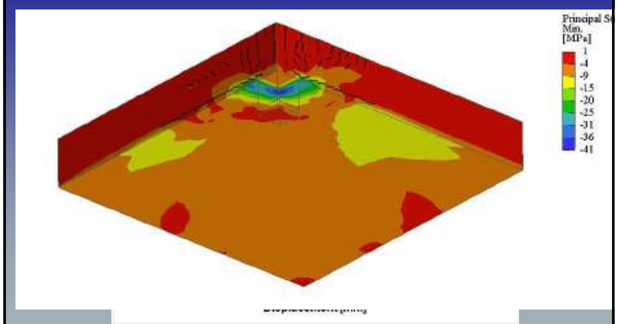
concrete: fracture-plastic constitutive law

steel: multilinear with hardening

bond-slip interface

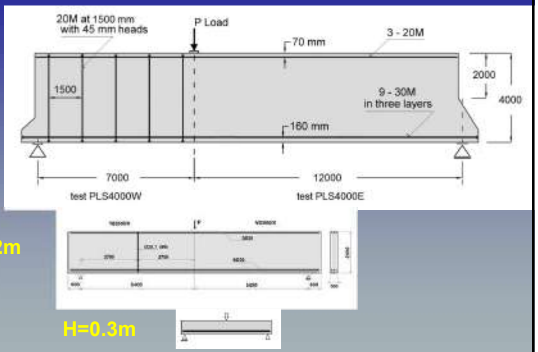
Punching shear tests by Guandalini, S. And Muttoni, A., EPFL, Lausanne

Punching shear



Prediction of shear strength test Torornto 2015

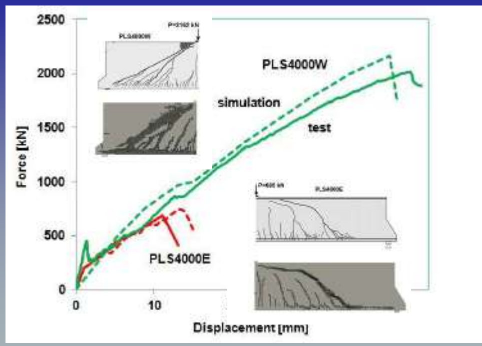
H=4m



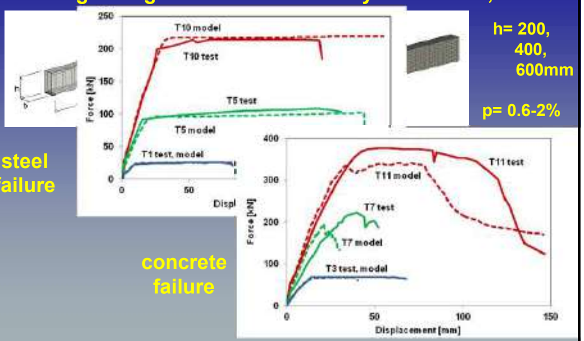
H=2m

H=0.3m

Shear strength predictions



Bending strength of normal beams by Debernardi, Torino



h= 200, 400, 600mm

p= 0.6-2%

steel failure

concrete failure

Safety factors in ULS due to model uncertainty NLFEA

Failure mode	μ_θ	V_θ	γ_{Rd}
Punching	0.971	0.076	1.16
Shear	0.984	0.067	1.13
Bending	1.072	0.052	1.01
All failure modes	0.979	0.081	1.16

CONCLUSIONS ULS

Model uncertaity is reflecting the knowledge comprised in numerical model.

Validation by experiments is required.

Safety factor for model uncertainty for ULS of all RC structural types and sizes

$$\gamma_{Rd} = 1.16$$

It is valid for ATENA models only

Uncertaities of crack models in SLS

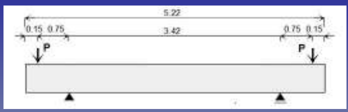
$$\theta_w = w_{experiment} / w_{model}$$

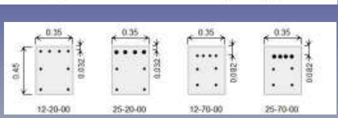
Model uncertainty θ_w is a random variable

μ_θ - mean uncertainty, model validation

V_θ - coefficient of variation, measure of uncertainty

Experiments by Alejandro Perez Caldentey





variable:

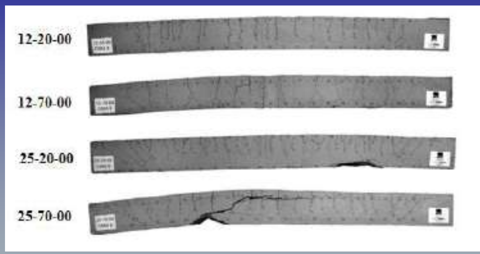
reinforcement Ø 12 and Ø 25mm

cover 20 and 70mm

fc = 27 Mpa

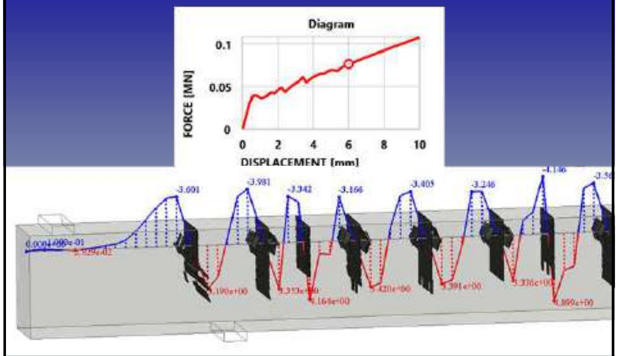
fyk = 500 MPa

Experimental crack patterns



Crack development

Bond stress



Model uncertainty of crack width

					Model uncertainty mean	V_θ
12-20-00	P [kN]	69.6	89.2	161.6		
	w_{mean} [mm]	0.231	0.305	1.233		
	w_{max} [mm]	0.245	0.328	1.560		
	$\theta_{w\,mean}$	0.711	0.797	0.513	0.67	0.22
	$\theta_{w\,max}$	1.510	1.250	0.621	1.13	0.41
12-70-00	P [kN]	51.1	60.7	100		
	w_{mean} [mm]	0.297	0.353	0.691		
	w_{max} [mm]	0.297	0.374	0.780		
	$\theta_{w\,mean}$	0.828	1.079	0.947	0.95	0.13
	$\theta_{w\,max}$	1.111	1.275	1.474	1.29	0.14
25-20-00	P [kN]	102.1	183.3	403		
	w_{mean} [mm]	0.062	0.126	0.280		
	w_{max} [mm]	0.068	0.150	0.420		
	$\theta_{w\,mean}$	1.971	1.178	1.259	1.47	0.30
	$\theta_{w\,max}$	2.916	1.913	1.638	2.16	0.31
25-70-00	P [kN]	56.7	101.7	298.8		
	w_{mean} [mm]	0.082	0.192	0.621		
	w_{max} [mm]	0.089	0.217	0.877		
	$\theta_{w\,mean}$	1.360	1.229	1.193	1.26	0.07
	$\theta_{w\,max}$	1.787	1.544	1.324	1.55	0.15
All specimens all load levels	$\theta_{w,mean}$	1.218	1.071	0.978	**1.09**	**0.35**
	$\theta_{w,max}$	1.831	1.496	1.264	**1.53**	**0.36**

Thank You for your attention!

STRUCTURAL RELIABILITY ANALYSIS IN AEROSPACE INDUSTRY

Frank Grooteman
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Solving real engineering problems taking into account uncertainties requires probabilistic methods that are robust (can handle multiple and complex limit-states), efficient (can be solved in a minimum number of simulations) and accurate in computing small probabilities. Many probabilistic methods have been proposed in literature over the past decades. Efficiency, accuracy and robustness are contradicting requirements and many methods lack one of these criteria making them less useful.

Two probabilistic methods developed by NLR will be briefly presented that have as much as possible the above characteristics. Moreover, a number of constraints will be presented related to aerospace problems. For instance, in aerospace industry the probability of failure is 10^-5 or less and in case of probabilistic fracture mechanics the limit-state function is discontinuous making it much harder to solve requiring a very robust probabilistic method. Apart from the cumulative probability of failure the hazard rate often is a required output which in many cases is much harder to compute.

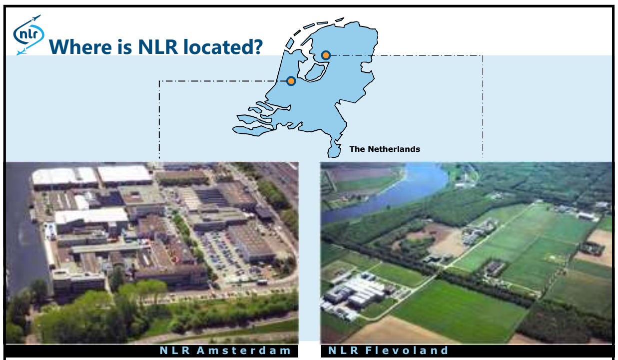
Dedicated to innovation in aerospace

Structural reliability analysis in aerospace industry

Frank Grooteman (frank.grooteman@nlr.nl)
Structural Integrity department
Netherlands Aerospace Centre (NLR)

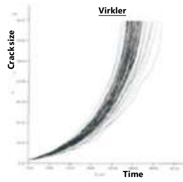
Contents

Where is NLR located?



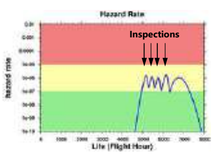
Introduction

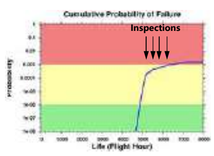
- Fracture mechanics is a very important design criteria of metal aircraft structure
- Considerable variation is observed in crack growth life time
  - Mainly caused by:
    - Scatter in initial flaw sizes
    - Scatter in loads
    - Scatter in (crack growth) material properties
- Variation in lifetime covered by:
  - Deterministic Damage Tolerance Analysis
    - (Large) initial flaw size + safety factor (2 to 3)
  - Probabilistic **Structural Risk Analysis**
    - Scatter taken into account by their distribution functions
    - Computes the probability of failure (reliability)
- Other application area for probabilistic analysis is composite structures
  - More scatter observed in properties than in metals



Introduction

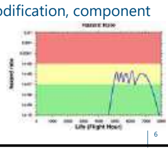
- Structural risk analysis (**SRA**)
  - An evaluation of a potential structural hazard and probability of failure
  - Hazard rate *h*(t) and/or probability of failure *F*(t) over time
- For new military aircraft SRA are mandatory
  - For example F-35, KF-X
  - Also applied more and more for existing aircrafts, e.g. F-16
  - High level description in MIL-STD-1530 and MIL-STD-882





MIL-STD-1530: Hazard rate

- MIL-STD-1530
  - Single Flight Probability of Failure (SFPoF)/Hazard rate *h*(t) $h(t) = \frac{f(t)}{1-F(t)}$
    - Probability of catastrophic failure happening in next flight
  - Threshold risk levels
    - SFPoF $\leq 10^{-7}$ is adequate, no action needed
    - 10^{-7} < SFPoF < 10^{-5} is undesirable
    - SFPoF > 10^{-5} is unacceptable
  - Values in between → risk mitigation measures must be taken
    - Operational restrictions, inspections, repair, modification, component replacement, aircraft retirement
  - Note: threshold risk levels are indicative



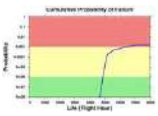
MIL-STD-882: Cumulative failure probability

POF CATEGORIZATION		SEVERITY*			
		CATASTROPHIC (1)	CRITICAL (2)	MARGINAL (3)	NEGLIGIBLE (4)
PROBABILITY	FREQUENT (A) $\geq 10^{-1}$	1	3	6	13
	PROBABLE (B) $\geq 10^{-2}$	2	5	9	16
	OCCASIONAL (C) $\geq 10^{-3}$	4	7	11	18
	REMOTE (D) $\geq 10^{-6}$	8	10	14	19
	IMPROBABLE (E) $< 10^{-6}$	12	15	17	20

HIGH	CAE Risk Acceptance HRI = 1 through 5/6
MEDIUM	PM Risk Acceptance HRI = 10 through 17
SERIOUS	PEO Level Risk Acceptance HRI = 6/7 through 9
LOW	Risk Acceptance As Directed HRI = 18 through 20

* Severity is the worst credible consequence of a hazard in terms of injury, property damage or effect on mission defined below:

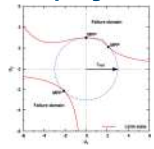
(1) **Catastrophic**: Class A (damage > $2M / fatality / permanent total disability / loss of aircraft)
(2) **Critical**: Class B ($500K < damage < $2M / permanent partial disability / hospitalization of 5 or more personnel)
(3) **Marginal**: Class C ($50K < damage < $500K / injury results in 1 or more lost workdays)
(4) **Negligible**: All other injury/damage less than Class C

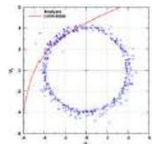


Adaptive Radial Based Importance Sampling (ARBIS)

- Based on Monte-Carlo Simulation outside a sphere (Harbitz, RBIS)
- Characteristics
  - Much more efficient than MCS
  - Accurate
    - Converges to the exact answer, in general
  - Very robust
  - β_{opt} unknown beforehand
    - Very efficient adaptive approach
  - NLR in-house developed method*
    - Implemented in RAP++

$$p_f = \iint_{G(\underline{u})\leq 0} f(\underline{u})\, d\underline{u} \overset{N_{sim}\to\infty}{=} \frac{N_{fail}}{N_{sim}}\left(1-\chi_n^2(\beta)\right)$$



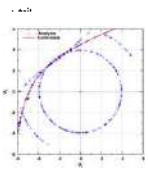


*Grooteman, F.P.; *Adaptive radial-based importance sampling method for structural reliability*, Structural Safety, Vol. 30, No. 6, pp. 533-542, 2008

Directional Simulation

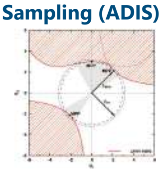
- Similar as MCS, but now random directions are sampled instead of points
- Characteristics
  - In general, much more efficient than MCS
    - When number of random variables < ±100
  - Accurate
    - Converges to the exact answer, in general
  - Very robust in general

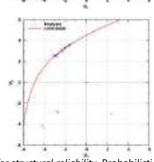
$$p_f = \iint_{G(\underline{u})\leq 0} f(\underline{u})\, d\underline{u} \overset{N_{sim}\to\infty}{=} \frac{1}{N_{sim}}\sum_{i=1}^{N_{sim}}\left(1-\chi_n^2(\beta_i)\right)$$



Adaptive Directional Importance Sampling (ADIS)

- Based on Directional Simulation
- Improved efficiency by means of Importance Sampling
  - Knowledge required of important regions
    - Limited pre-sampling
    - Create a Response Surface for sampling in unimportant directions
    - Performing exact simulations in important regions (grey area, small β)
      - Update Response Surface with this information
- Characteristics
  - Accurate, efficient and robust in general
  - NLR in-house developed method*
    - Implemented in RAP++





*Grooteman, F.P.; *An Adaptive Directional Importance Sampling method for structural reliability*, Probabilistic Engineering Mechanics, Vol. 26, pp. 134-141, 2011

Quantitative Comparison Probabilistic Methods

Method	Accuracy	Efficiency	Robustness
MCS (LHS)	High	Low	High
DS (ARBIS)	High	Medium-low	High
ADIS	High-medium	High-medium	High-medium
SORM	Medium-low	High-medium	Low
FORM	Low	High	Low

- Accuracy = Error in probability of failure
- Efficiency = Required number of (expensive) deterministic analyses
- Robustness = Ability to handle complex limit-state(s)
  - Multiple failure points, multiple failure functions, noisy/discontinous limit state, …
- FORM/SORM cannot handle complex limit-states

Final remarks

- Robustness
  - Failure function can be (highly) discontinuous in fracture mechanics problem
    - For instance, in case of variable amplitude loading where failure occurs on the same high load cycle for a RV parameter range
- Accuracy
  - Low probability of failure, up to 10^{-9} or below for civil aircraft, 10^{-6} for military aircraft
- Besides cumulative probability $F(t)$ of failure, hazard rate $h(t)$ is often required
- Probability of failure over time required not only final value
- Number of important/significant random variables often limited (< 15)
  - Should be determined first by a relatively cheap sensitivity analysis
  - Number pf RVs can be higher in case of random field discretization (e.g. composites), but correlation is (often) unknown
  - Data gathering for each RV will otherwise become even more costly
    - Lack of data already biggest problem!

Final remarks

- Curse-of-dimensionality in case of meta/response surface models
  - Even for a Fractional Factorial Designs
  - Popular Kriging much worse, requiring (many) internal points as well
    - No good choice in general
- Lack of accuracy in case of meta/response surface models
  - Small error in meta model yields a large error (order(s) of magnitude) in PoF
  - Use of response surface only to determine important limit-state(s)
- The more efficient a reliability method is, the more dependence on previous knowledge in each step, the less the possible parallelisation of the algorithm
  - High Performance Computing with many (> 1000) processors becomes cheaper and cheaper
  - Crude MCS or DS the (near) future?
  - Commercial software license issues!

Computation of cumulative failure probability

$$P_f = P\left(G(\underline{u}) \leq 0\right) = \iint_{G \leq 0} f(\underline{u}) d\underline{u}$$

- Solution of the integral equation is complex
  - Multi-dimensional integral equation
  - Joint Probability Density Function f(u) unknown in general
  - Limit-state G(u)=0 unknown in *explicit form* in general
    - Requires evaluation of an external code, e.g. finite element tool, crack growth tool, …
- Multiple evaluations of the failure function G required
  - Search for an efficient probabilistic method that requires a minimum number of G-function evaluations (deterministic analyses)
  - In general, small probabilities ($< 10^{-3}$) for engineering problems
- Robust, efficient and accurate probabilistic method needed

RELIABILITY ANALYSIS IN GEOTECHNICAL PRACTICE – EXPERIENCES AND CHALLENGES

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Reliability analysis in geotechnical engineering – Experiences and challenges –

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Delft University of Technology

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Delft University of Technology

Outline

1. WHAT IS SPECIAL ABOUT GEOTECHNICAL ENGINEERING?
2. RECENT PRACTICAL APPLICATIONS, INCLUDING BAYESIAN UPDATING
3. SUMMARY STATE OF PRACTICE AND CHALLENGES

Schweckendiek ('state of practice')

4. RFEM FOR SLOPE STABILITY
5. SUBSET SIMULATION WITH RANDOM FIELDS
6. INSIGHTS IN FAILURE MODES WITH HETEROGENEITY

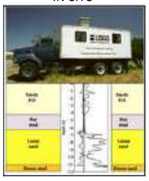
Van den Eijnden ('state of the art')

What is special about geotech?

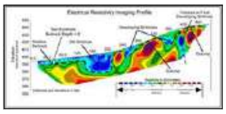
- NATURAL MATERIAL (NO QUALITY CONTROL) *-> HETEROGENEITY*
- LIMITED SITE INVESTIGATION *-> (EPISTEMIC) UNCERTAINTY*

LAB

IN-SITU



GEOPHYSICS

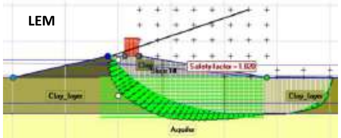


Accuracy
Spatial coverage

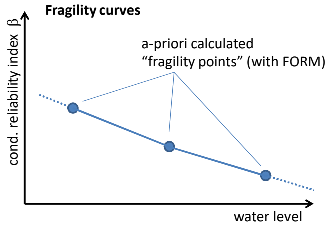
→ PARTICULARLY AMENABLE TO PROBABILISTIC TREATMENT!

Dike slope stability

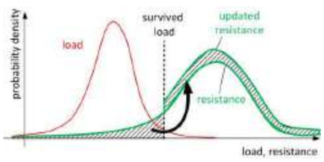
LEM



Fragility curves



Reliability updating for flood defenses - Slope stability -

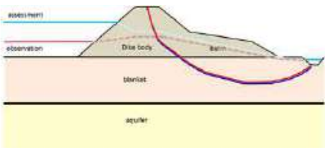


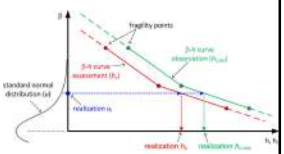
Direct method (Bayes' rule):

$$P(F|\varepsilon) = \frac{P(F \cap \varepsilon)}{P(\varepsilon)}$$

failure evidence

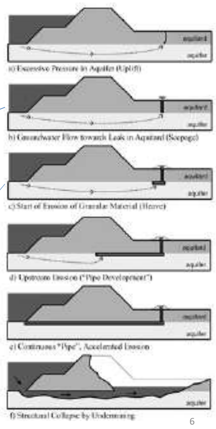
Even smaller probability than Pf!





Reliability updating for slope stability of dikes - Approach with fragility curves (background report).
http://publications.deltares.nl/1230090_033.pdf

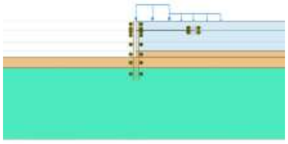
Reliability updating for flood defenses - Piping -



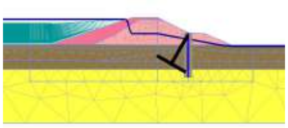
Schweckendiek, T., Vrouwenvelder, A.C.W.M., & Calle, E.O F. (2013). Updating Piping Reliability with Field Performance Observations. Structural Safety (47), 13-23.

Ground-structure interaction with FEM

RETAINING WALLS / QUAY WALLS



STRUCTURALLY REINFORCED DIKES



EXPERIENCES
- main structural failure modes can typically be tackled with FORM (wall, anchor)
- simpler approaches often too inaccurate or only for low dimensions (e.g. PEM)
- MCS not an option due to computation time
- Directional Sampling works if model not too heavy (and implicit treatment spatial variability)

MAIN COMUTATIONAL CHALLENGES
- (strong) non-linearities (e.g. transition elastic -> plastic soil behaviour; uplift conditions)
- Overall instability: no calculation response in failure domain

State-of-practice in a nutshell

- EC0 AND EC7 PROVISIONS (INCL. OBSERVATIONAL METHOD)
- FEW COUNTRIES SEEM TO EXPLOIT THIS
- RAPIDLY GROWING INTEREST IN NL (DUE TO FLOOD DEFENSES)
- WHAT ARE THE PRACTICAL GEOTECH APPLICATIONS WE DO SEE?

high-reliability installations (e.g. GATE LNG-terminal)

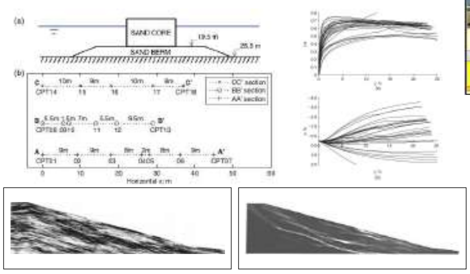
natural hazards (hurricanes, landslides etc.)

design optimization (e.g. offshore foundations)

WE NEED:
1. ROBUST AND EFFICIENT COMPUTATIONAL METHODS
2. 'INTERPRETABLE' RESULTS!

From uncertainty in layers to spatial variability

EARLY EXAMPLE: THE NERLERK UNDERWATER BERM FAILURE

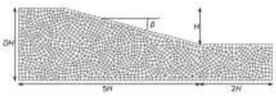


Hicks MA, Onisiphorou C. Stochastic evaluation of static liquefaction in a predominantly dilative sand fill. *Geotechnique*, **55**(2), 123-133 (2005)

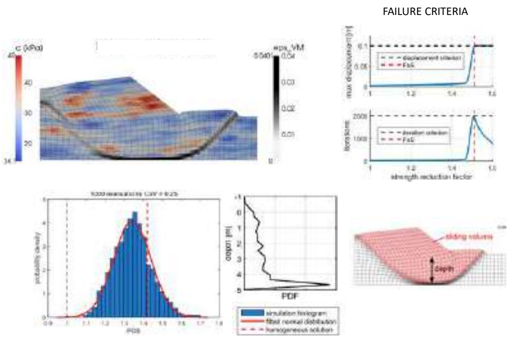
RFEM for slope reliability analysis (1)

- FINITE ELEMENT MODEL
  - NO CONCEPTUAL COMPROMISE ON PHYSICAL MODEL

FOR NOW:
- TRESCA MATERIAL MODEL
  - LINEAR ELASTIC – PERFECT PLASTIC
  - c AS MATERIAL PARAMETER
- STATIAL VARIABILITY CHARACTERISED BY COVARIANCE FUNCTION
  - STATIONARY RANDOM FIELDS



RFEM for slope reliability analysis (2)

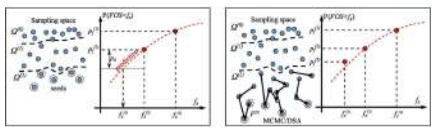


Subset simulation and RFEM (1)

- CHARACTERISATION OF RANDOM FIELD Z USING COVARIANCE MATRIX $\mathbf{C}$:

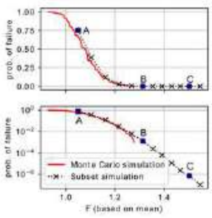
$$Z = \mathbf{C}^{1/2}U, \qquad \mathbf{C}^{1/2} = \Phi\Lambda^{1/2}\Phi^{\mathrm{T}}$$

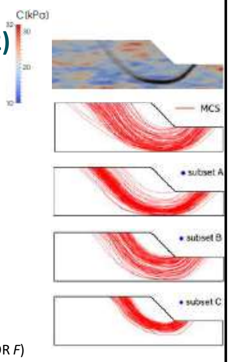
- LIMIT STATE FUNCTION IN U-SPACE: N-DIMENSIONAL
  - POSSIBLE REDUCTION OF PARAMETERS
- ADDRESS FAILURE DOMAIN USING SUBSET SIMULATION DRIVEN BY F_S



Eijnden AP van den, Hicks MA. Efficient subset simulation for evaluating the modes of improbable slope failure. *Computers and Geotechnics*, **88**, 267-280 (2017)

Subset simulation and RFEM (2)



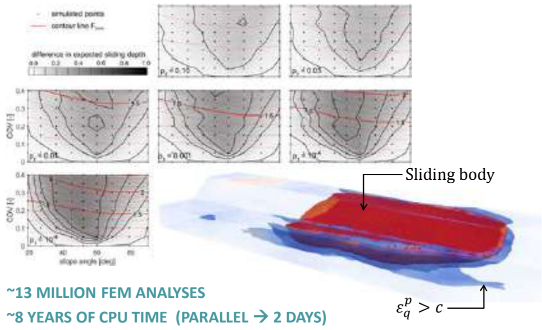


IMPROBABLE SLOPE FAILURE:

- SLOPES WITH KNOWN MEAN STRENGTH (FACTOR *F*)
- FAILING DUE TO UNCERTAINTY IN SPATIAL VARIABILITY (CONSTANT *COV*)

→PREDOMINANTLY STABLE SLOPES SHOW SHALLOW MODES OF FAILURE

Application in parametric analyses (2D)



- ~13 MILLION FEM ANALYSES
- ~8 YEARS OF CPU TIME (PARALLEL → 2 DAYS)

Summary RFEM

1. A MAJOR CHALLENGE IS PROPER MODELLING OF HETEROGENEITY AND EPISTEMIC UNCERTAINTY (NOT ONLY COMPUTATIONAL)
2. SIMULATIONS REMAIN COMPUTATIONAL VERY DEMANDING
3. DIFFICULT TO VERIFY CONVERGENCE OF SOLUTION (POSTERIOR)
4. DETAILED 3D ANALYSIS OF SPATIALLY VARIABLE SLOPES AT SMALL FAILURE PROBABILITIES REMAINS A CHALLENGE
5. SUBSET SIMULATION MOST PROMISING SO FAR…

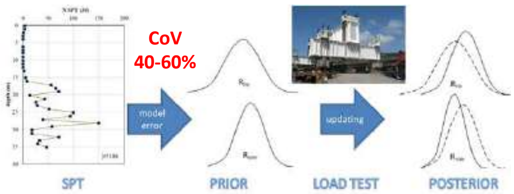
Deltares
Enabling Delta Life

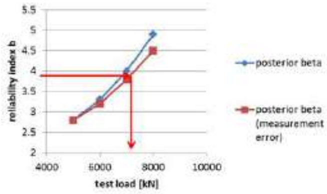
TU Delft

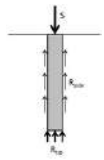
TIMO SCHWECKENDIEK
DELTARES
DELFT UNIVERSITY OF TECHNOLOGY

BRAM VAN DEN EIJNDEN
DELFT UNIVERSITY OF TECHNOLOGY

Foundation piles







RELIABILITY ASSESSMENTS OF CONCRETE STRUCTURES BASED ON NONLINEAR FINITE ELEMENT ANALYSES: HOW TO CODIFY DESIGN METHODS?

Max Hendriks
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TU Delft, Netherlands & NTNU, Norway

Reliability assessments of concrete structures based on Nonlinear Finite Element Analyses: how to codify design methods?

Reporting from action group 8 contributing to the *fib* Model Code 2020

Max Hendriks – TU Delft, Netherlands & NTNU, Norway
TNO Workshop *Computational challenges in the reliability assessment of engineering structures*, 24 January 2018, Delft

In this presentation

- Introducing the *fib* and the Model Code
- Issues
- Way forward

What is the *fib* Model code 2020?

- Short name: *fib* MC2020
- Update of the *fib* MC2010 with added data on "existing concrete structures"
- Will serve as a basis for future codes for concrete structures
- For national and international code committees, practitioners and researchers

fib Action Groups

- Focussing on a specific topic/section with in the MC2020
- Action group «AG8»: focussing on section «7.11 Verifications assisted by numerical simulations»

fib Action Group AG8

- 20 members
- A "core team"
  - Giorgio Monti (co-convenor)
  - Diego Allaix
  - Morten Engen (technical secretary)
  - Max Hendriks (convenor)

fib AG8 Current status of the work

- Wishes for the MC2020 text of 7.11 have been investigated.
- Working on specifications for the text.

«ISSUES»

Model uncertainties

- Defined as the ratio of <u>observed load resistance</u> and finite element predictions of the load resistance.
- That is, the main application field is estimating the load resistance of a concrete structure.

Model uncertainties

1. There is not one nonlinear finite element approach. Many approaches exist with different choices for the
   - Kinematic equations
   - Constitutive equations
   - Equilibrium methods & conditions
2. Very often the approaches have not documented explicitly

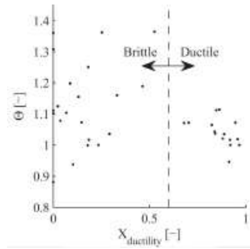
Model uncertainties

3. Some finite element models are like "virtual experiments" and simulate failure. Others model "only" the force redistributions and use a "simple" failure criterion.

Model uncertainties

4. The application field of the models is wide.
5. The model uncertainty depends on the type of failure mode. That is, it depends on the "brittleness" of the failure.

Model uncertainties



M. Engen et al. / Structural Safety 64 (2017) 1–8

Model uncertainties

Table 2-2: Statistical properties of the modelling uncertainty per failure mode

Failure mode	Mean	CoV
Bending	0.97	0.04
Flexural shear in beams	1.01	0.08
Shear in slabs	1.39	0.10
All	1.15	0.19

Rijkswaterstaat technisch document 1016-2:2017, 2017

Model uncertainties

6. Mainly based on lab experiments which are always idealizations of actual structures

7. Hard to unravel from other (material) uncertainties

Model uncertainties

8. Sometimes based on “between-model uncertainty” with 1 experimental outcome and multiple model approaches:

$$\theta_{1,i} = \frac{R_{\text{exp}}}{R_{\text{NLFEA},i}}$$

(It describes the obtained uncertainty in the prediction if a model was selected randomly)

Morten Engen, PhD thesis NTNU, 2017

Reliability methods

- Semi-probabilistic «safety formats» based on limited calibrations.

«WAY FORWARD»

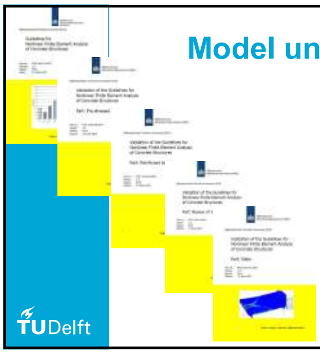
Model uncertainties

1. Based on a “within-model uncertainty” adopting a fixed modelling approach

$$\theta_{3,i} = \left(\frac{R_{\text{exp}}}{R_{\text{NLFEA}}}\right)_i$$

Morten Engen, PhD thesis NTNU, 2017

Model uncertainties



2. Use fixed = documented modelling approaches.

E.g. based on guidelines
–or–
on advices from the software program developers (?)

Rijkswaterstaat technisch document 1016-1,2,3:2017, 2017

Model uncertainties

3. Provide values per "type of failure mode" and per "level of model calibration" (???)
4. Provide the possibility to determine the model uncertainty of a certain modelling approach for a certain application area (?)

Reliability methods

1. Provide methods based on response surfaces (???)
   - Attractive from an engineering point of view
   - Can be interpreted
2. Provide methods based on <u>calibrated</u> semi-probabilistic approaches

Concluding remark

- Work to do between now and 2020