innovation for life

TNO Workshop: Computational Challenges in the Reliability Assessment of Engineering Structures
Delft, The Netherlands

Hyper-spherical Importance Sampling and Extrapolation for High Dimensional Reliability Problems

Junho SONG

Professor, Ph.D.
Department of Civil & Environmental Engineering
Seoul National University, S. Korea

Ziqi WANG*/王子琦

Assistant Professor, Ph.D.
Earthquake Engineering Research & Test Center
Guangzhou University, China

Seoul National University
Department of Civil&Environmental Engineering

SSRG
Structural & System Reliability Group

广州大学工程抗震研究中心
Earthquake Engineering Research & Test Center of Guangzhou University

High dimensional Euclidean space

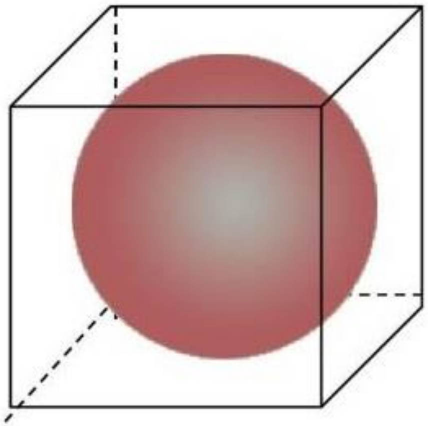
Volume Explosion

In n-dimensional space, consider a hypersphere inscribed in a hypercube

$$V_{hypersphere} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} R^n$$

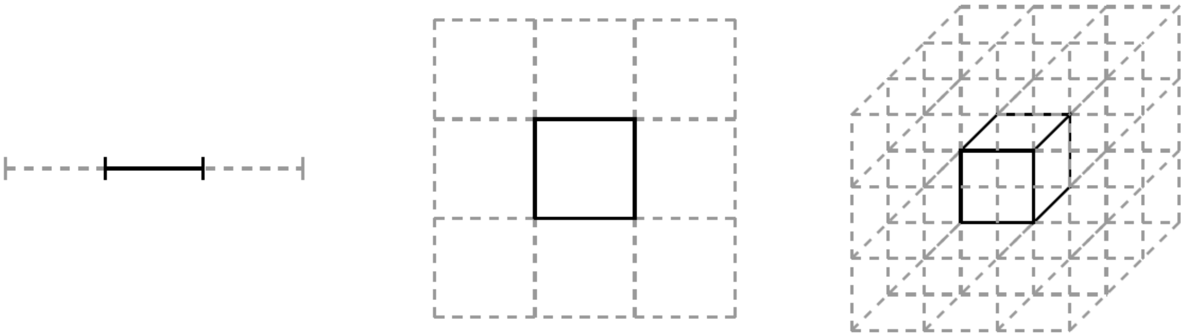
$$V_{hypercube} = (2R)^n$$

$$\frac{V_{hypersphere}}{V_{hypercube}} = \frac{\pi^{n/2}}{2^n\Gamma\left(\frac{n}{2}+1\right)} \to 0, n \to +\infty$$



Volume Concentration

Volume tends to distribute in the 'tails'



Betancourt (2017)

High dimensional probability space

There may exist a typical set

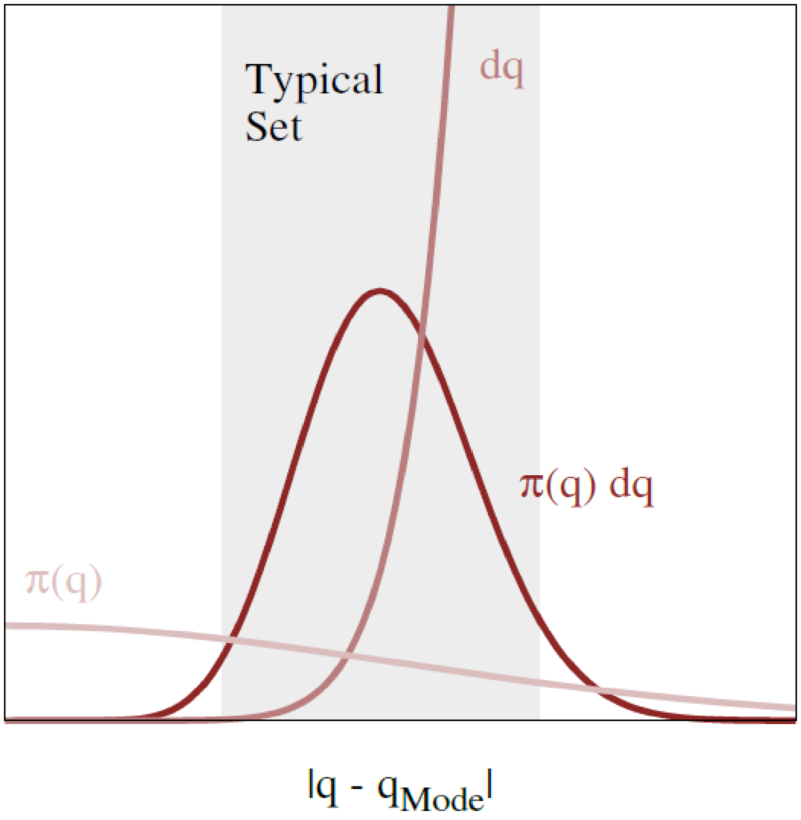
In n-dimensional space, consider the probability

$$\Pr\left(\boldsymbol{q} \in \Omega\right) = \int_{\boldsymbol{q} \in \Omega} \pi(\boldsymbol{q})\, d\boldsymbol{q}$$

Betancourt (2017)

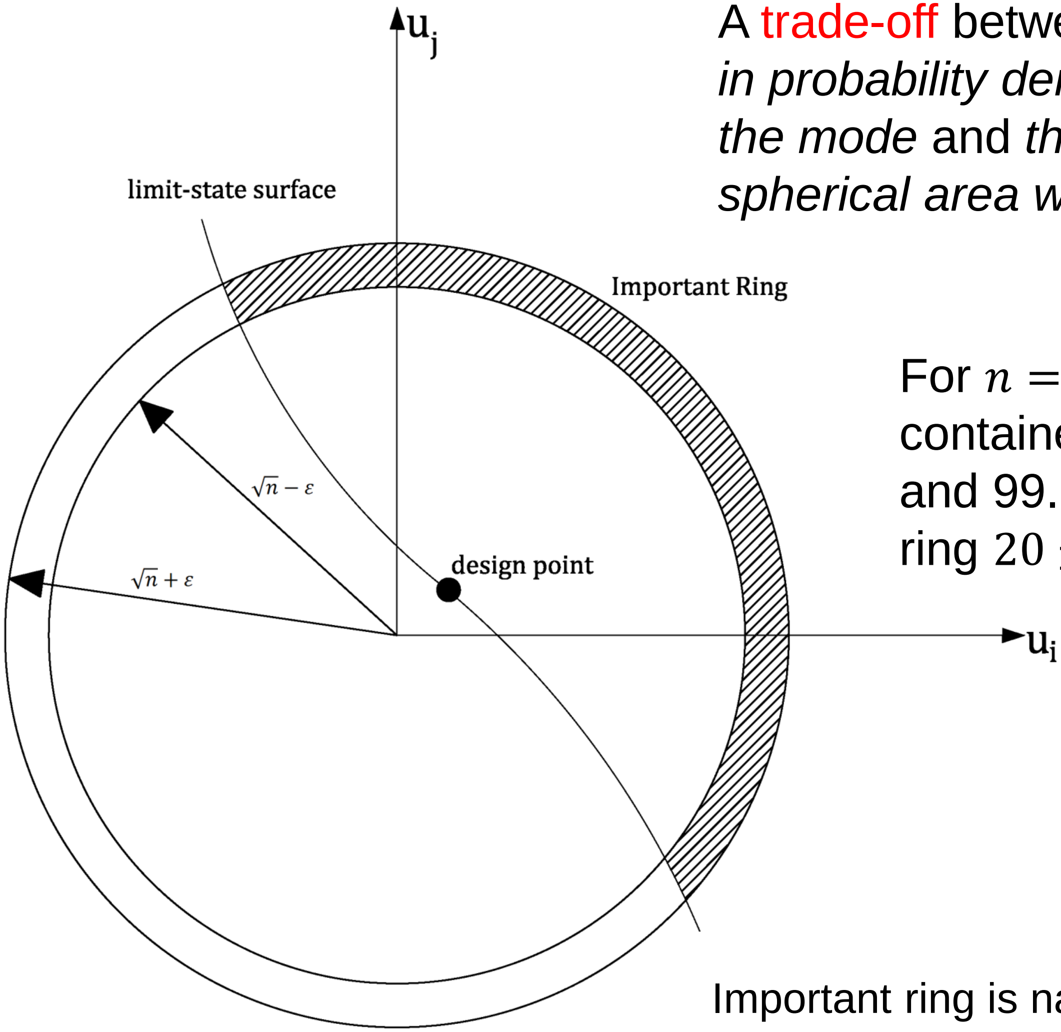
PDF $\pi(\boldsymbol{q})$ concentrates around its mode,

$d\boldsymbol{q}$ is much larger away from the mode



High dimensional standard normal space

The typical set is a hyper-ring

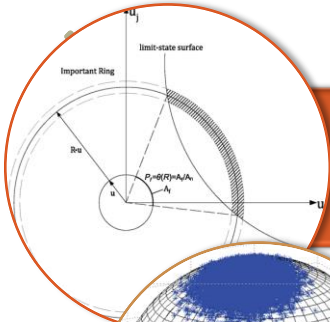


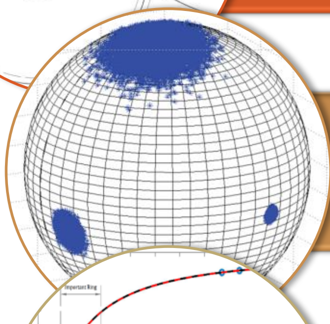
A trade-off between *the exponentially decrease in probability densities with the distance from the mode* and *the exponentially increase in the spherical area with the distance from the mode*

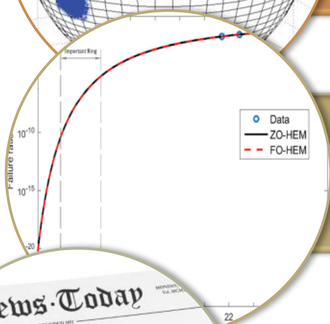
For $n = 400$, 95% probability is contained within the ring 20 ± 1, and 99.99% is contained within the ring 20 ± 2.

Important ring is named by Katafygiotis and Zuev (2008)

Contents

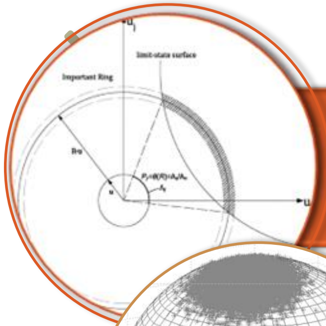

limit-state surface
Important Ring

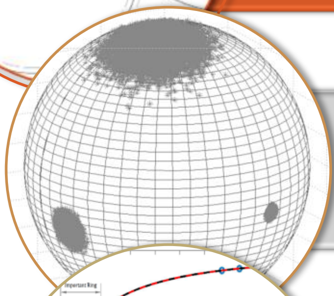


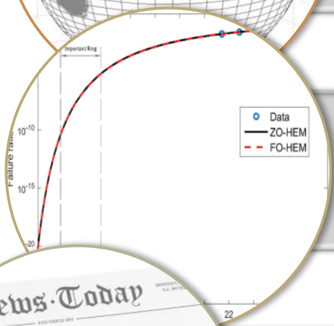

Data
ZO-HEM
FO-HEM


News·Today
WHAT'S
NEXT?

Contents


Limit-state surface
Important Ring



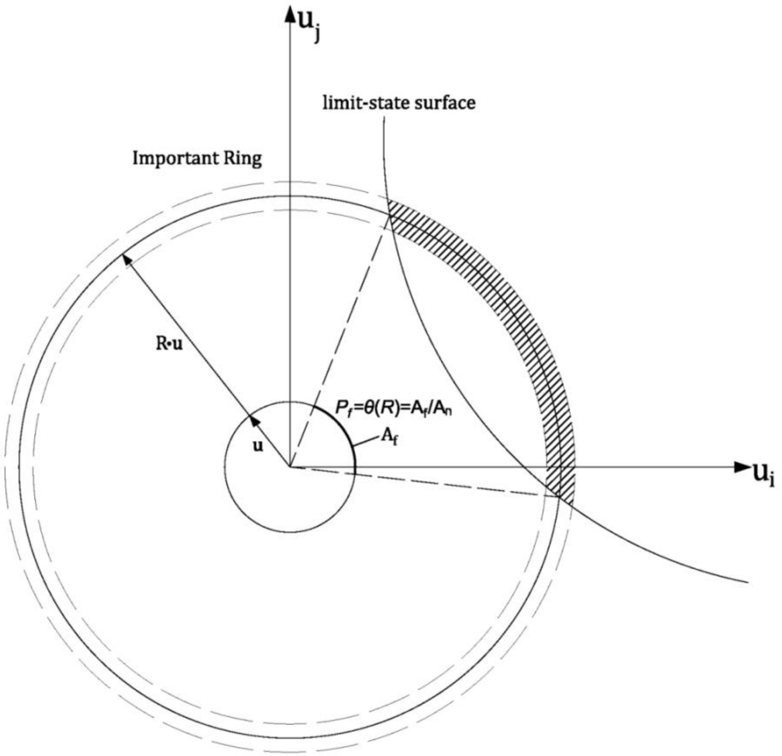

Data
ZO-HEM
FO-HEM


News·Today
WHAT'S
NEXT?

Hyper-spherical formulation

$$P_f = \int_0^\infty \theta(r) f_\chi(r) dr \cong \frac{1}{M} \sum_{i=1}^{M} \theta(r_i)$$

where $\theta(r) = A_f(r)/A_n$, $A_n = \frac{n\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$

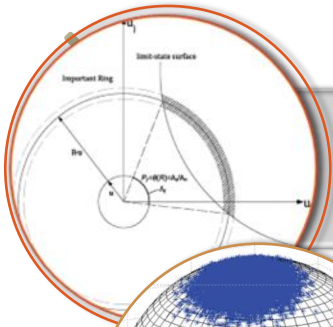


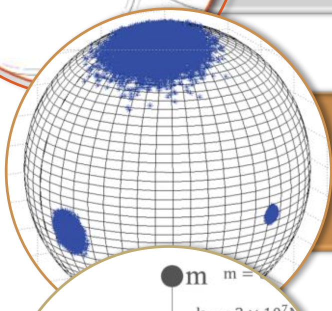
- Valid for any dimensions
- Especially convenient for high dimensional problems

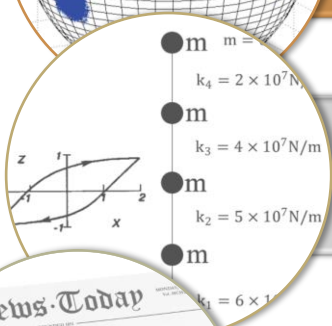
r_i drawn from $f_\chi(r)$ is likely to have $r_i \in [\sqrt{n} - \varepsilon, \sqrt{n} + \varepsilon]$.

Variation of $\theta(r_i)$ with r_i (drawn from $f_\chi(r)$) is expected to be small

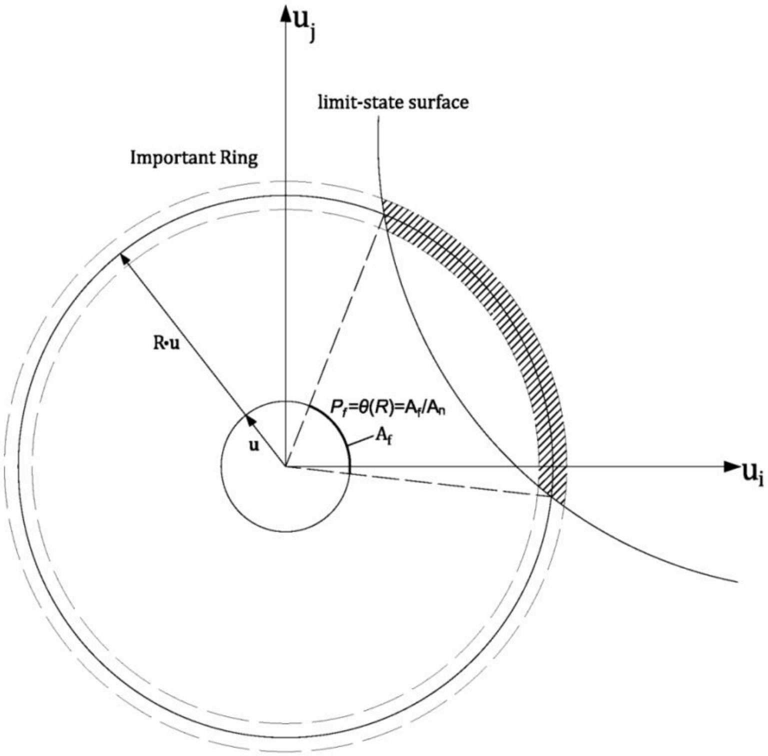
Contents







Hyper-spherical formulation based importance sampling



$$P_f = \int_0^{\infty} \theta(r) f_{\chi}(r) dr \cong \frac{1}{M} \sum_{i=1}^{M} \theta(r_i)$$

Construct an IS density to estimate $\theta(r_i)$

$$\theta(r_i) = \int \frac{I_{r_i}(r_i \bar{\mathbf{u}})}{A_n} d\bar{\mathbf{u}}$$

$$= \int \frac{I_{r_i}(r_i \bar{\mathbf{u}})}{A_n f_{IS}(\bar{\mathbf{u}})} f_{IS}(\bar{\mathbf{u}}) d\bar{\mathbf{u}}$$

$$\cong \frac{1}{N} \sum_{j=1}^{N} \frac{I_{r_i}(r_i \bar{\mathbf{u}}_j)}{A_n f_{IS}(\bar{\mathbf{u}}_j)}$$

Finally, the IS

$$P_f \cong \frac{1}{N \cdot M} \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{I_{r_i}(r_i \bar{\mathbf{u}}_j)}{A_n f_{IS}(\bar{\mathbf{u}}_j)}$$

where r_i drawn from $f_{\chi}(r)$, $\bar{\mathbf{u}}_j$ drawn from $f_{IS}(\bar{\mathbf{u}})$

Von Mises-Fisher Mixture as the IS density

Wang, Z., and Song J.(2016). Cross-entropy-based adaptive importance sampling using von Mises–Fisher mixture for high dimensional reliability analysis. *Structural Safety*. 59: 42-52.

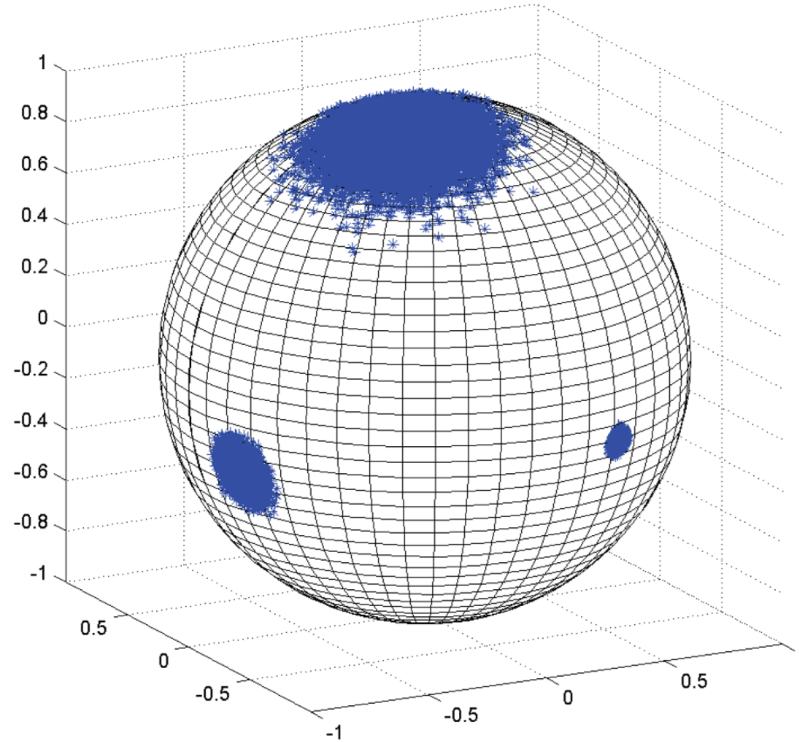
- **Sampling by "von Mises-Fisher Mixture" model**

$$f_{\mathrm{vMFM}}(\bar{\mathbf{u}};\mathbf{v}) = \sum_{k=1}^{K} \alpha_k\, f_{\mathrm{vMF}}(\bar{\mathbf{u}};\mathbf{v}_k)$$

where $\sum_{k=1}^{K} \alpha_k = 1$, $\alpha_k > 0 \text{ for } \forall k$

$$f_{\mathrm{vMF}}(\bar{\mathbf{u}}) = c_d(\kappa) e^{\kappa \boldsymbol{\mu}^T \bar{\mathbf{u}}}$$

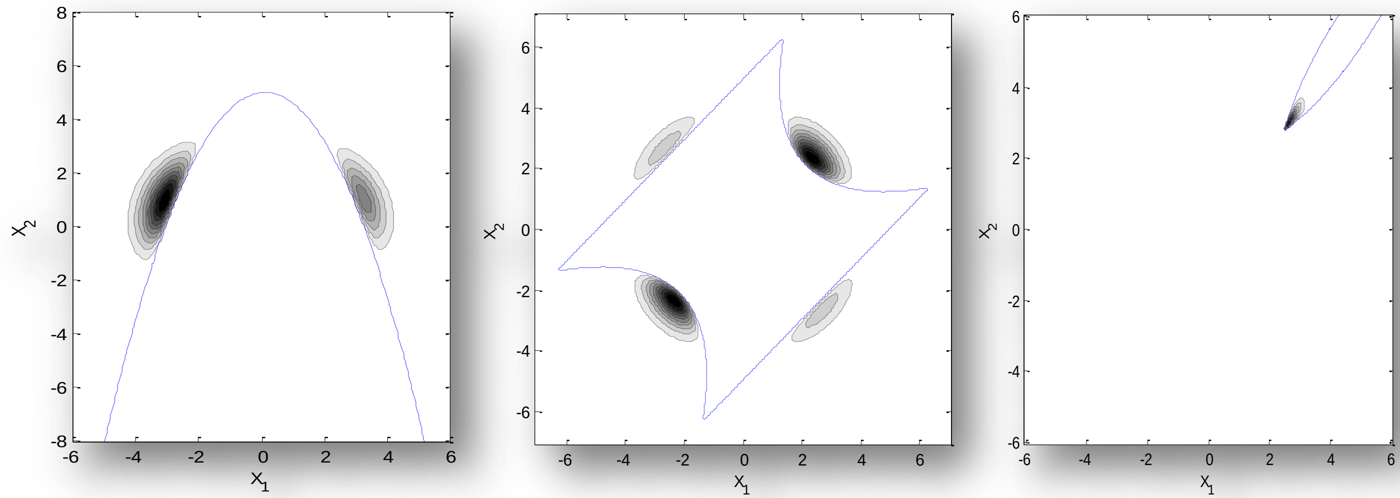
- κ: concentration parameter
- $\boldsymbol{\mu}$: mean direction
- α_k: weight for the *k*-th vMF



How can we find parameters of the vMFM model?

"Best" importance sampling density

$$p^*(\mathbf{x}) = \frac{|H(\mathbf{x})|}{\int |H(\mathbf{x})| d\mathbf{x}} = \frac{I(\mathbf{x}) f_X(\mathbf{x})}{P_f}$$



- Can't use directly… if we already know P_f, we do not need MCS or IS.
- Still helpful for improving efficiency, if $h(\mathbf{x})$ is chosen in order to have **a shape similar to that of $I(\mathbf{x})f_X(\mathbf{x})$**

Adaptive importance sampling by minimizing cross entropy

Kullback-Leibler "Cross Entropy" (CE)

$$D(p^*, h) = \int p^*(\mathbf{x}) \ln p^*(\mathbf{x}) d\mathbf{x} - \int p^*(\mathbf{x}) \ln h(\mathbf{x}) d\mathbf{x}$$

- "Distance" between "best" IS density $p^*(\mathbf{x})$ and current one $h(\mathbf{x})$
- One can find a good $h(\mathbf{x})$ by minimizing Kullback-Leibler CE, i.e.

$$\arg\min_{\mathbf{v}} D(p^*, h(\mathbf{v})) = \arg\min_{\mathbf{v}} \left[\int p^*(\mathbf{x}) \ln p^*(\mathbf{x}) d\mathbf{x} - \int p^*(\mathbf{x}) \ln h(\mathbf{x}; \mathbf{v}) d\mathbf{x} \right]$$

$$= \arg\max_{\mathbf{v}} \int p^*(\mathbf{x}) \ln h(\mathbf{x}; \mathbf{v}) d\mathbf{x}$$

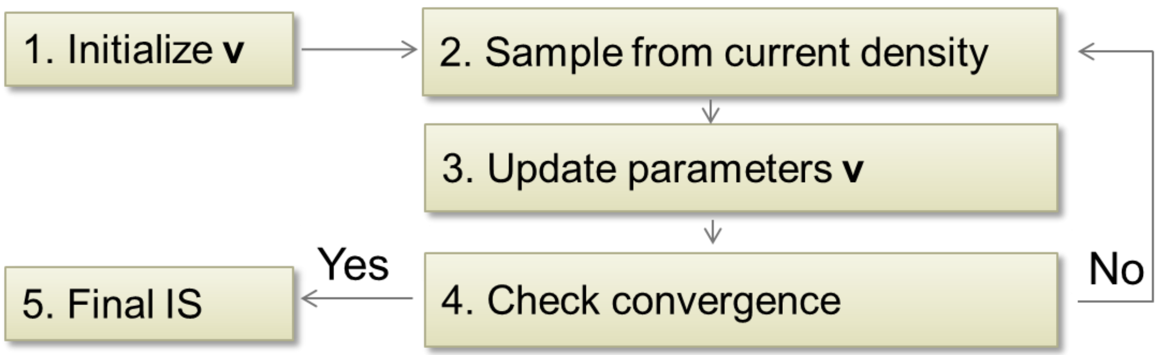
$$= \arg\max_{\mathbf{v}} \int I(\mathbf{x}) f_{\mathrm{X}}(\mathbf{x}) \ln h(\mathbf{x}; \mathbf{v}) d\mathbf{x}$$

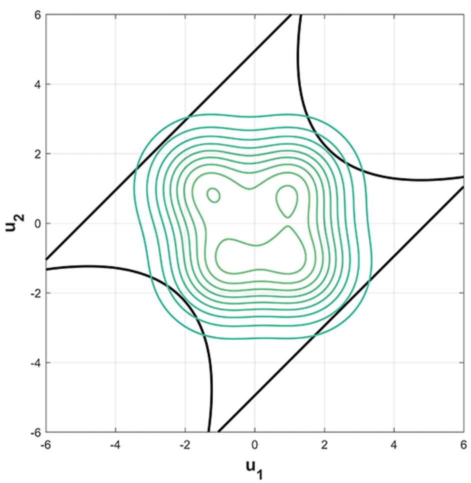
- Finds the optimal values of the distribution parameter(s) **v approximately by small-size pre-sampling**, then performs final importance sampling
- Rubinstein & Kroese (2004) used **uni-modal parametric distribution** for $h(\mathbf{x};\mathbf{v})$ and provided **updating rules** to find optimal **v** through sampling

CE-AIS with Gaussian Mixture (Kurtz & Song 2013)

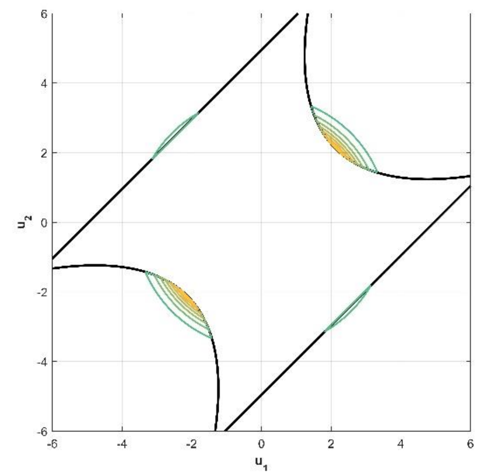
Kurtz, N., and Song J. (2013). Cross-entropy-based adaptive importance sampling using Gaussian mixture. *Structural Safety*. 42:35-44.

- **CE-AIS-GM Algorithm** $h(\mathbf{x};\mathbf{v}) = \sum_{k=1}^{K} \pi_k N(\mathbf{x} \mid \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$



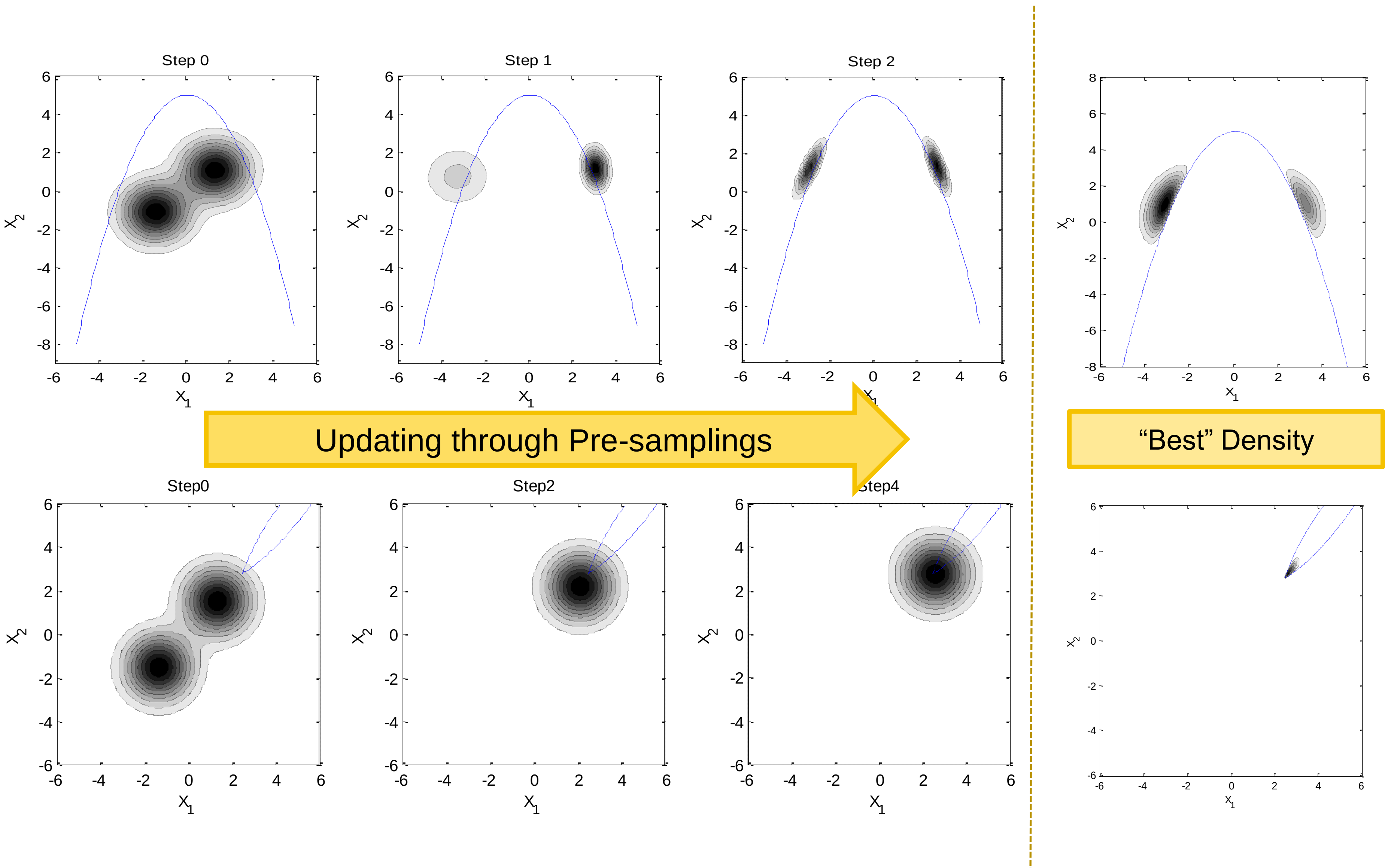


Near Optimal Density



Optimal Density

CE-AIS with Gaussian Mixture (Kurtz & Song 2013)



Parameter estimation for vMFM model

$$\alpha_k = \frac{\sum_{\boldsymbol{j}=1}^{N} I_{r_i}(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j; \mathbf{w}) \Upsilon_{\boldsymbol{j}}(z_k)}{\sum_{\boldsymbol{j}=1}^{N} I_R(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j; \mathbf{w})}$$

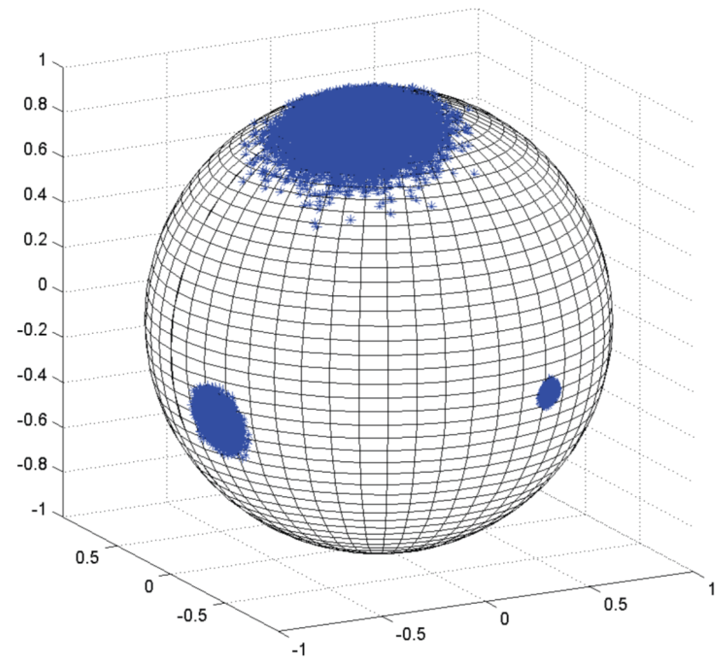
$$\boldsymbol{\mu}_k = \frac{\sum_{i=1}^{N} I_{r_i}(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j; \mathbf{w}) \Upsilon_{\boldsymbol{j}}(z_k) \bar{\mathbf{u}}_j}{\left\| \sum_{i=1}^{N} I_{r_i}(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j; \mathbf{w}) \Upsilon_{\boldsymbol{j}}(z_k) \bar{\mathbf{u}}_j \right\|}$$

$$\kappa_k \cong \frac{\xi n - \xi^3}{1 - \xi^2}$$

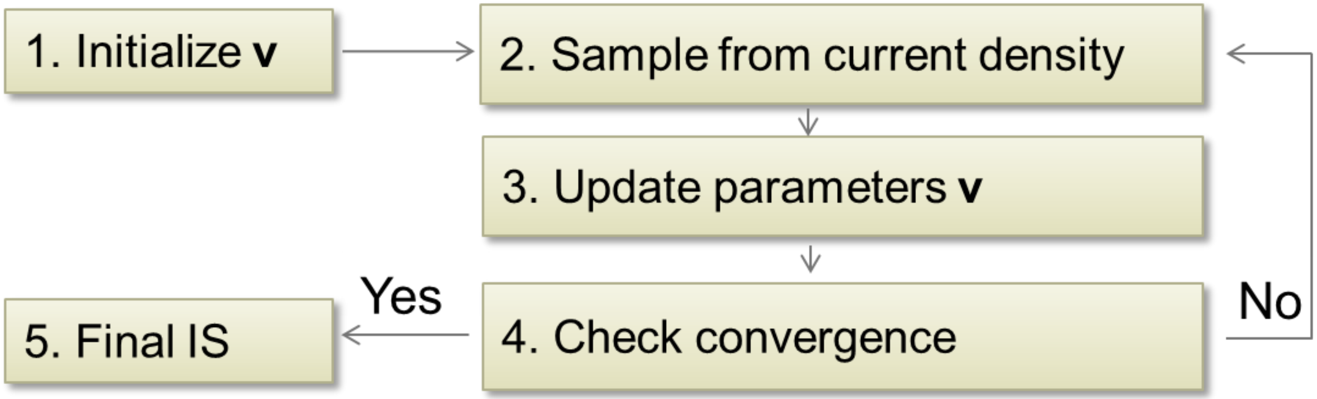
where

$$\xi = \frac{\left\| \sum_{i=1}^{N} I_{r_i}(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j; \mathbf{w}) \Upsilon_{\boldsymbol{j}}(z_k) \bar{\mathbf{u}}_j \right\|}{\sum_{i=1}^{N} I_{r_i}(\bar{\mathbf{u}}_j) W(\bar{\mathbf{u}}_j; \mathbf{w}) \Upsilon_{\boldsymbol{j}}(z_k)}$$

$$\Upsilon_{\boldsymbol{j}}(z_k) = \frac{\alpha_k f_{\mathrm{vMF}}(\bar{\mathbf{u}}_j; \mathbf{v}_k)}{\sum_{k=1}^{K} \alpha_k f_{\mathrm{vMF}}(\bar{\mathbf{u}}_j; \mathbf{v}_k)}$$



Procedures of Hyper-spherical importance sampling using vMFM



1. Pre-sampling to obtain near-optimal (i.e. minimum CE) vMFM sampling density using updating rules
2. Perform the final IS on hyper-spheres with radius drawn from the $f_\chi(r)$

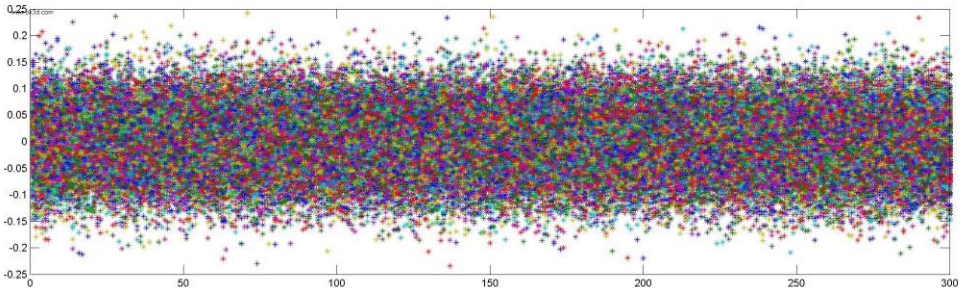
Example 1: Series system reliability in high-dimension

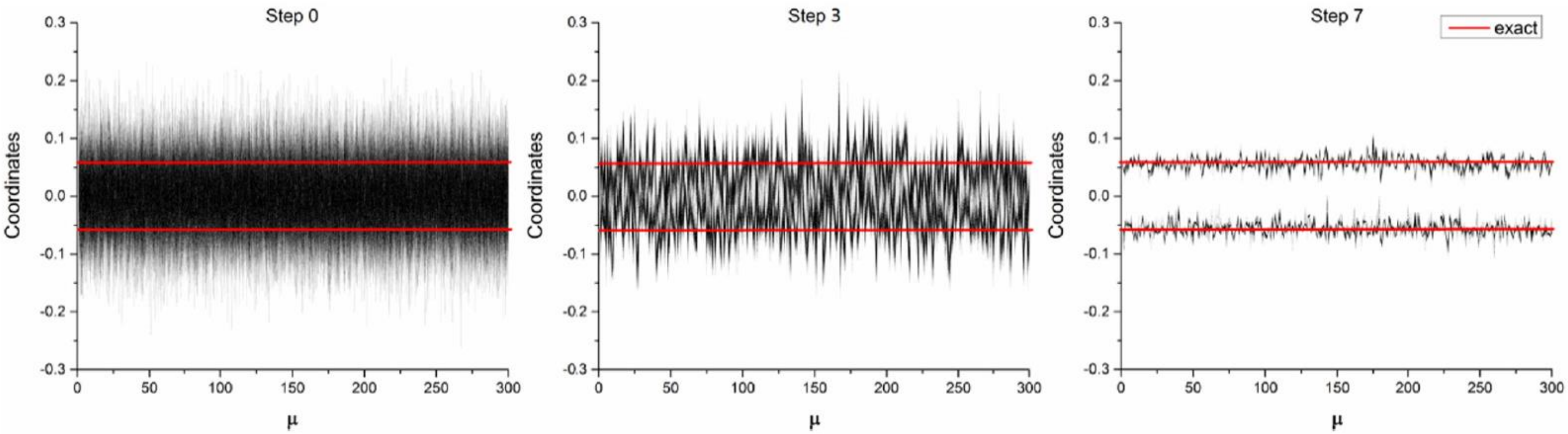
$$G_1(\mathbf{u}) = \beta_1\sqrt{n} - \sum_{i=1}^{n} \mathbf{u}_i \,, G_2(\mathbf{u}) = \beta_2\sqrt{n} + \sum_{i=1}^{n} \mathbf{u}_i$$

System failure domain: $G_1(\mathbf{u}) \leq \mathbf{0} \cup G_2(\mathbf{u}) \leq \mathbf{0}$

$\beta_1 = \beta_2 = 3.5, n = 300$

Updating of mean directions:





Example 2: Nonlinear random vibration analysis of MDOF system

- **Discrete representation of stochastic process representing ground acceleration (in frequency domain)**

$$\ddot{U}_g(t) = \sum_{j=1}^{n/2} \sigma_j [\mathrm{u}_j \cos(\omega_j t) + \hat{\mathrm{u}}_j \sin(\omega_j t)]$$

where

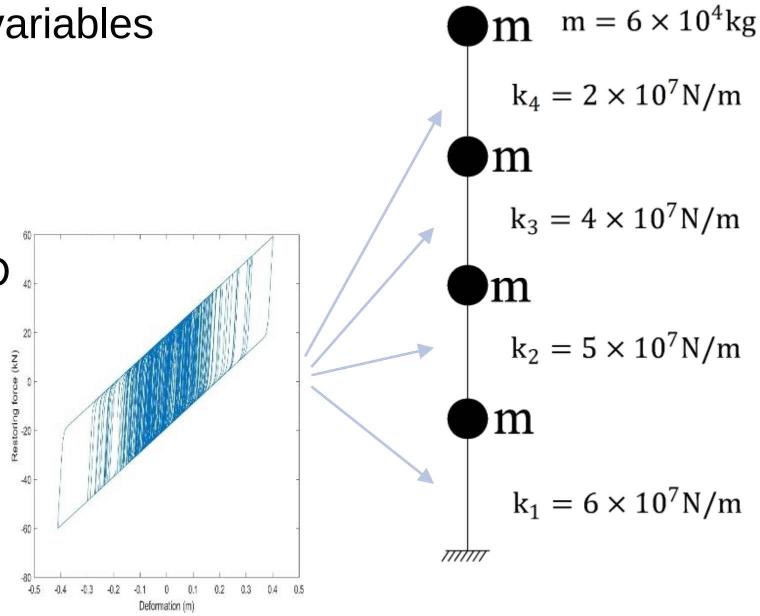
u_j, $\hat{\mathrm{u}}_j$: independent standard normal random variables

ω_j: discretized frequency points

$\sigma_{\boldsymbol{j}} = \sqrt{2S(\omega_j)\Delta\omega}$

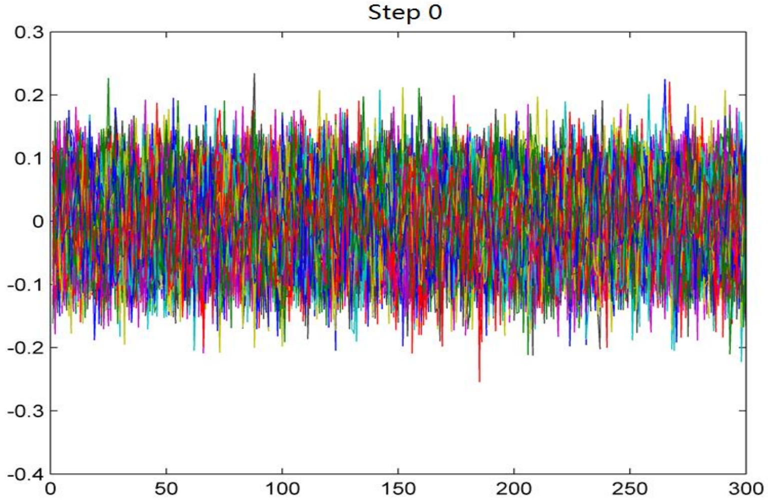
$S(\omega_j)$: two-sided power spectrum density/PSD

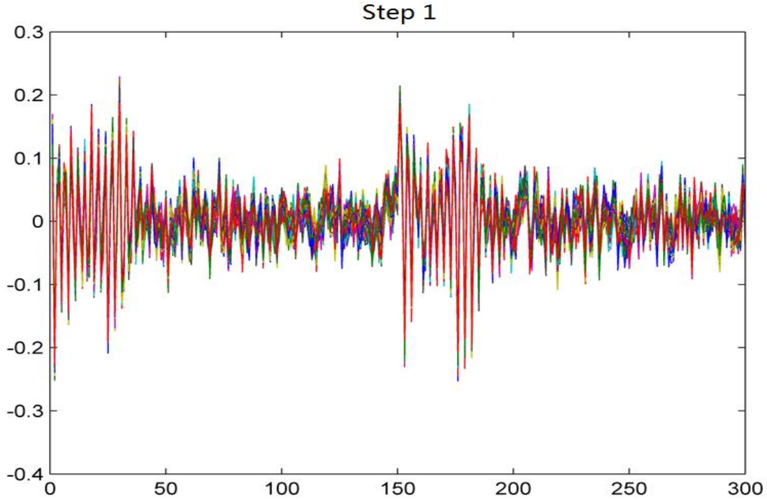
$\Delta\omega$: frequency step size



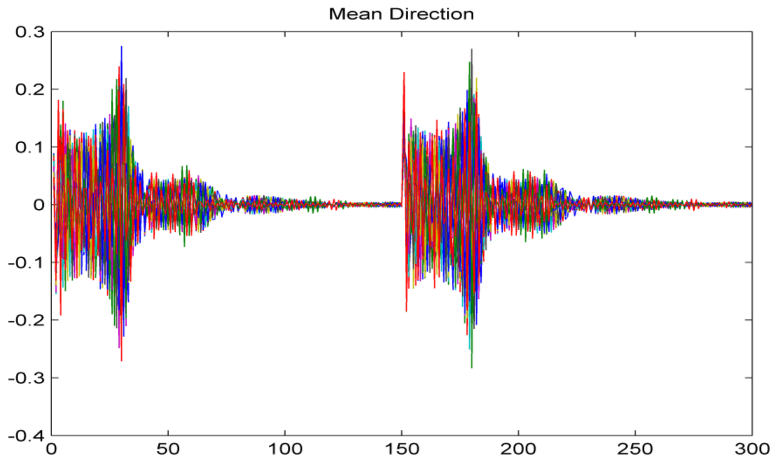
Example 2: Updating of vMFM

- **Instantaneous failure**

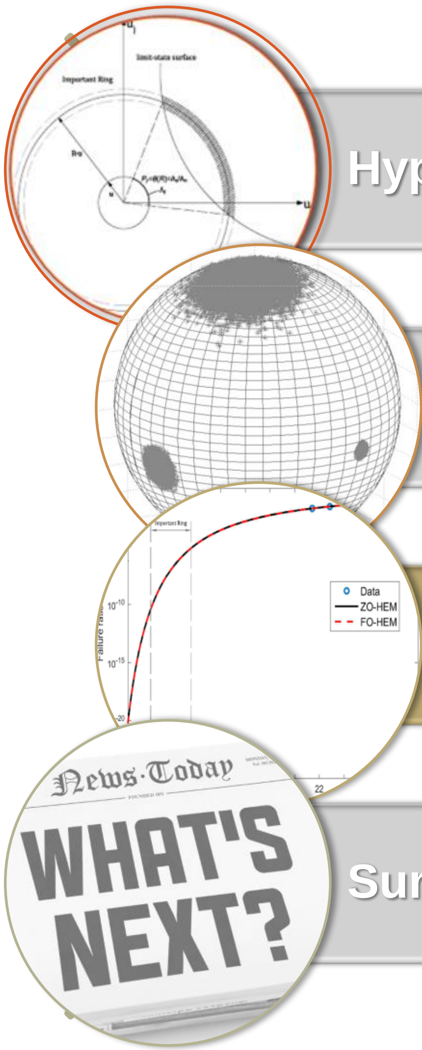




- **First-passage failure (series system)**



Contents


Limit-state surface
Important Ring
Data
ZO-HEM
FO-HEM
News·Today
WHAT'S NEXT?

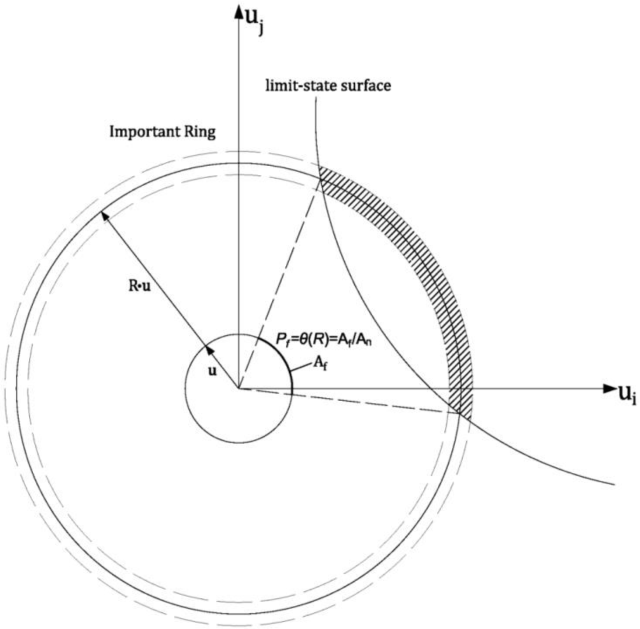
Hyper-spherical formulation based extrapolation

$$P_f = \int_0^\infty \theta(r) f_\chi(r) dr$$

Build an extrapolation method via writing $\theta(r)$ as $\theta(r) \cong \hat{\theta}(r, \mathbf{v})$

$$P_f = \int_0^\infty \theta(r) f_\chi(r) dr$$

$$\cong \int_{\sqrt{n}-\varepsilon}^{\sqrt{n}+\varepsilon} \hat{\theta}(r, \mathbf{v}) f_\chi(r) d\,r$$



Observe that $\theta(r)$ grows larger if r increases, given the safe domain is star-shaped with respect to the origin

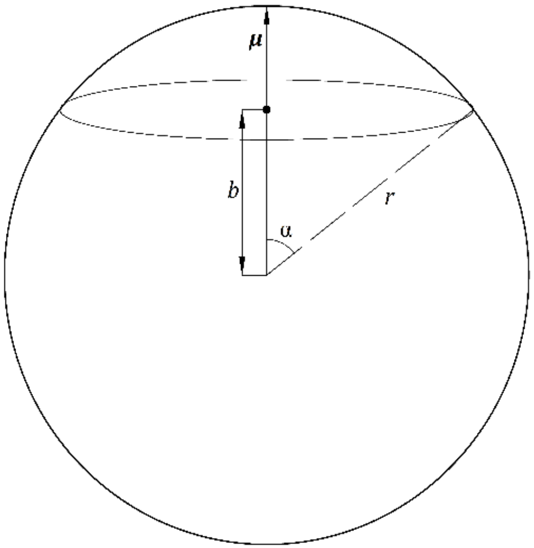
Concept of the extrapolation:

- Find $\mathbf{v}$ of $\hat{\theta}(r, \mathbf{v})$ given $\theta(r)$ estimated from large radius r
- Estimate P_f using the hyper-spherical formulation

Model for failure ratio $\hat{\theta}(r, \mathbf{v})$

Wang, Z., and Song J. (2018). Hyper-spherical extrapolation method (HEM) for general high dimensional reliability problems. Structural Safety, 72: 65–73.

$$\theta_{cap}(r, \alpha) = \frac{A_{cap}(r, \alpha)}{A_n(r)} = \frac{1}{2} B_{sin^2\alpha}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$



$B_{sin^2\alpha}(\cdot)$ is a regularized incomplete beta factor

$$\hat{\theta}(r, \alpha_k, K) = \sum_{k=1}^{K} \theta_{cap,k}(r, \alpha_k) = \frac{1}{2} \sum_{k=1}^{K} B_{sin^2\alpha_k}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

Considering the dependence of α_k on r

$$\hat{\theta}(r, b_k, K) = \frac{1}{2} \sum_{k=1}^{K} B_{1-\left[\frac{b_k(r)}{r}\right]^2}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

Assume $b_k(r)$ does not change dramatically with r

- *Zeroth-order hyper-spherical extrapolation method* (ZO-HEM):
$$b_k(r) = b_k$$
- *First-order hyper-spherical extrapolation method* (FO-HEM):
$$b_k(r) = a_k r + b_k$$

Probability estimation using HEM

- **ZO-HEM:**

$$P_f \cong \sum_{k=1}^{K} \Phi(-b_k)$$

- **FO-HEM:**

$$P_f \cong \frac{1}{2}\int_{\sqrt{n}-\varepsilon}^{\sqrt{n}+\varepsilon} \sum_{k=1}^{K} B_{1-\left(a_k+\frac{b_k}{r}\right)^2}\left(\frac{n-1}{2},\frac{1}{2}\right) f_\chi(r)\,d\,r$$

Procedures of HEM

- Select a sequence of m radii r_i, $i = 1, \ldots, m$, $r_i \in [r_{low}, r_{up}]$

- For each r_i, compute the failure ratio $\hat{\theta}(r_i)$

- Given $\hat{\theta}(r_i)$, compute optimal values of b_k and K in for ZO-HEM, or a_k, b_k and K for FO-HEM, so that the error function $\sum_{i=1}^{m} w_i \left[\log \hat{\theta}(r_i) - \log \theta(r_i)\right]^2$ is minimized, where w_i is a weight that puts more emphasis on more reliable data

- Compute the failure probability using CDF of standard normal distribution or numerical integration

Example 1: Series system reliability in high-dimension

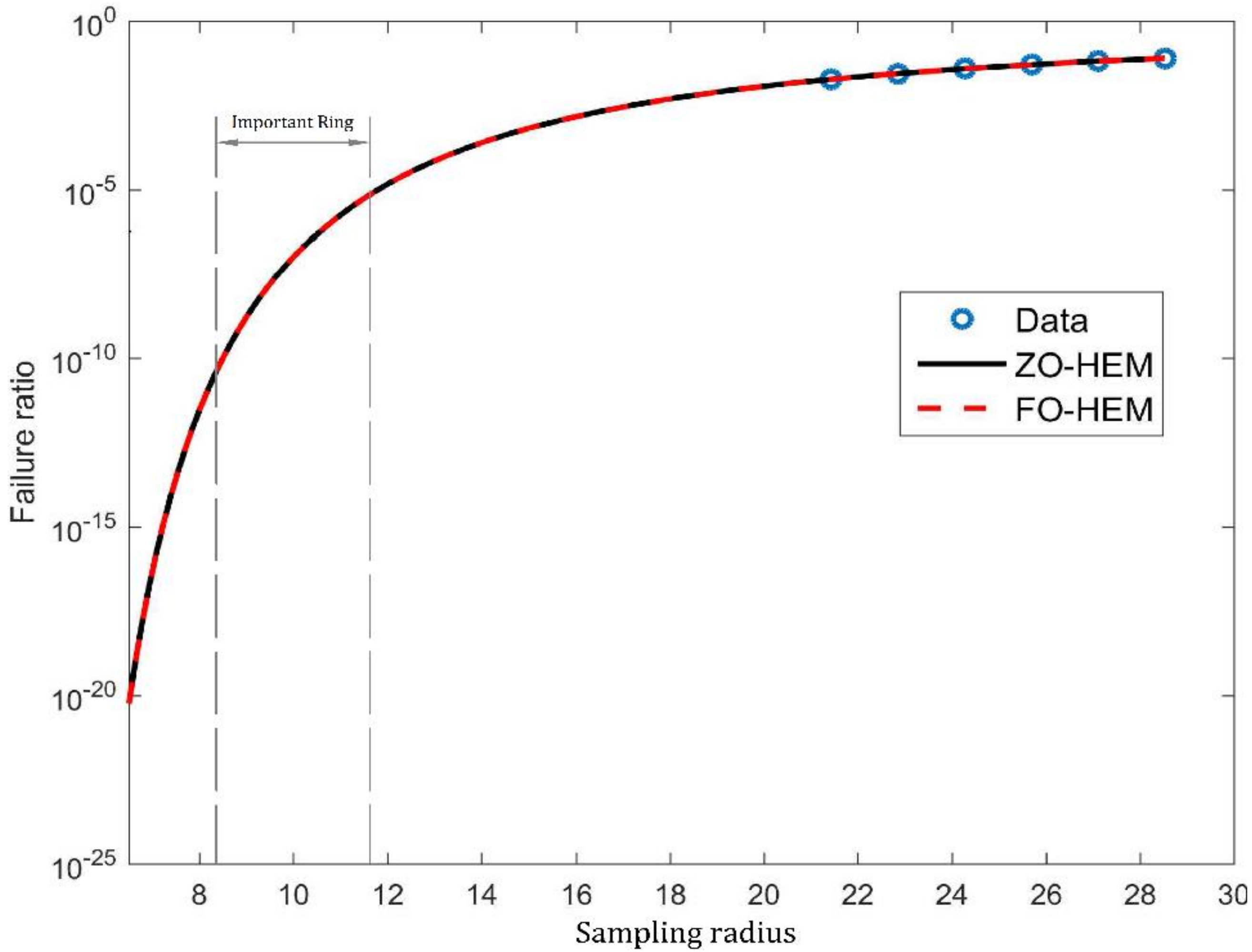
$$G_1(\mathbf{u}) = \beta_1\sqrt{n} - \sum_{i=1}^{n} \mathbf{u}_i \,, G_2(\mathbf{u}) = \beta_2\sqrt{n} + \sum_{i=1}^{n} \mathbf{u}_i$$

System failure domain: $G_1(\mathbf{u}) \leq \mathbf{0} \cup G_2(\mathbf{u}) \leq \mathbf{0}$

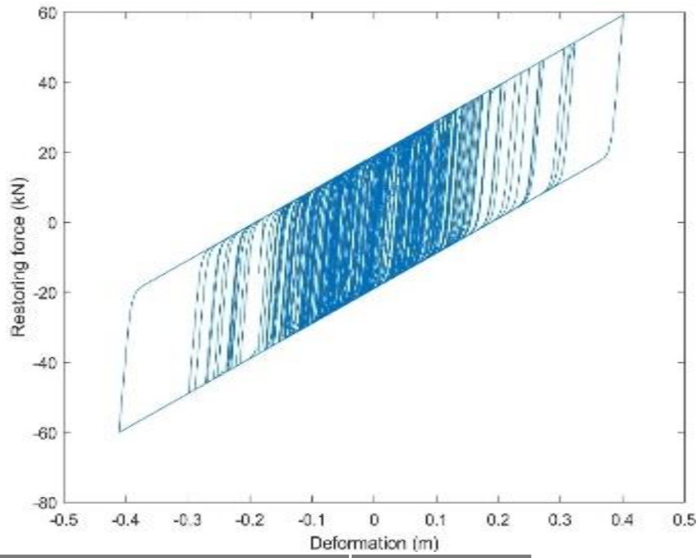
β_0	ZO-HEM			FO-HEM			Exact
	$\hat{\beta}$	c.o.v	Error (%)	$\hat{\beta}$	c.o.v	Error (%)	β
3.0	2.784	0.051	0.07	2.800	0.053	0.65	2.782
3.5	3.328	0.022	0.51	3.338	0.058	0.82	3.311
4.0	3.820	0.019	-0.33	3.846	0.043	0.33	3.833
4.5	4.366	0.009	0.36	4.381	0.025	0.71	4.350
5.0	4.906	0.052	0.86	4.894	0.051	0.59	4.865

Example 1: Series system reliability in high-dimension

$\theta(r)$ versus r curves for $\beta_0 = 5.0$



Example 2: Nonlinear random vibration analysis of SDOF system

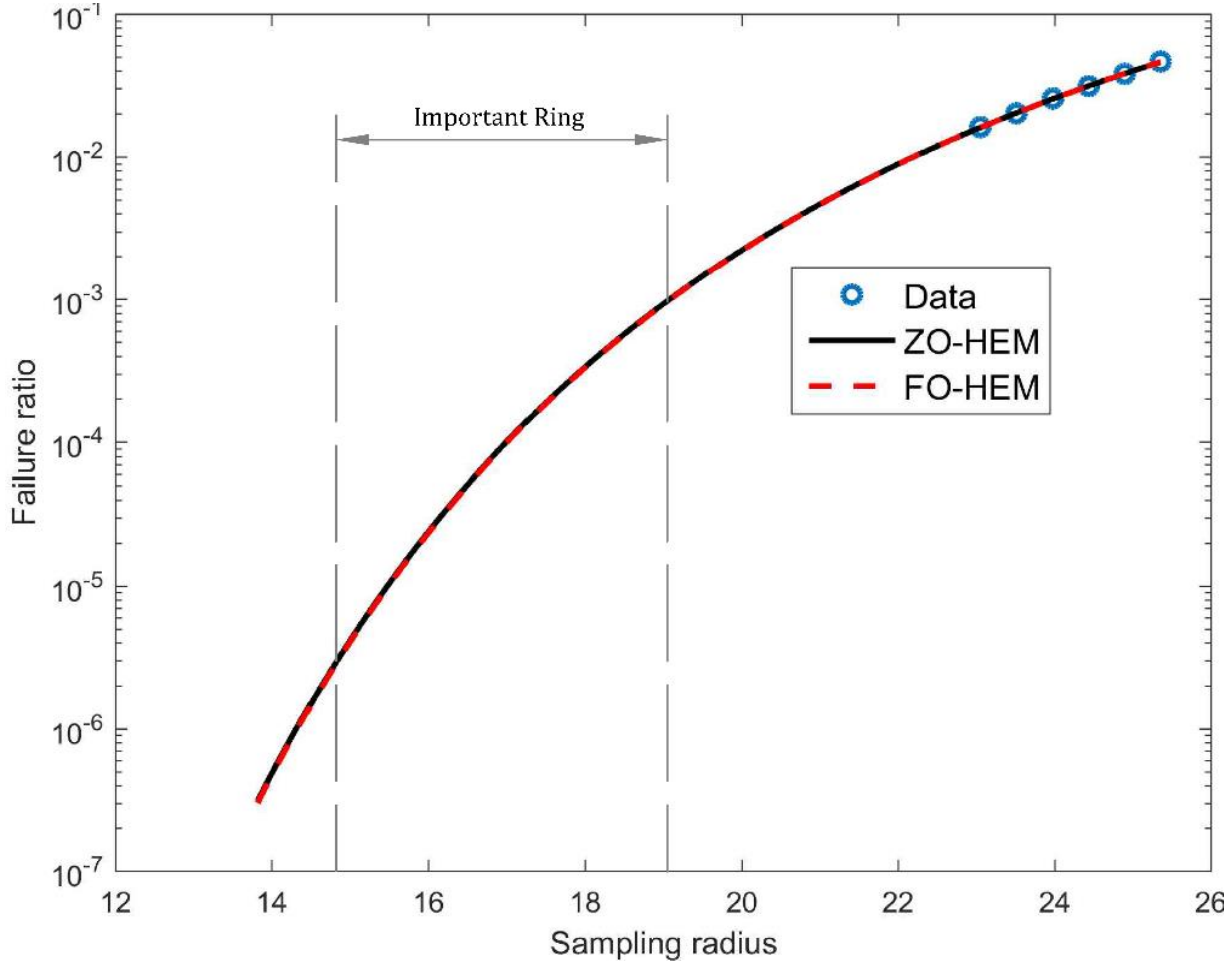


SDOF Bouc-Wen oscillator subjected to white noise

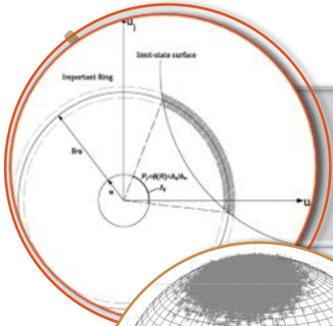
Threshold (m)	ZO-HEM			FO-HEM			Exact
	$\hat{\beta}$	c.o.v	Error (%)	$\hat{\beta}$	c.o.v	Error (%)	β
0.08	2.480	0.025	-2.95	2.518	0.043	-1.48	2.556
0.09	2.953	0.035	-2.72	2.971	0.048	-2.13	3.036
0.10	3.401	0.031	-3.92	3.475	0.037	-1.84	3.540

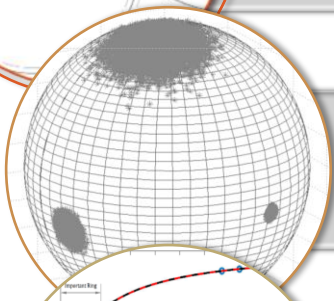
Example 2: Nonlinear random vibration analysis of SDOF system

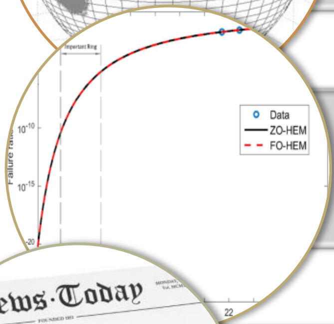
$\theta(r)$ versus r curves for 0.10 (m) threshold



Contents


Limit-state surface
Important Ring



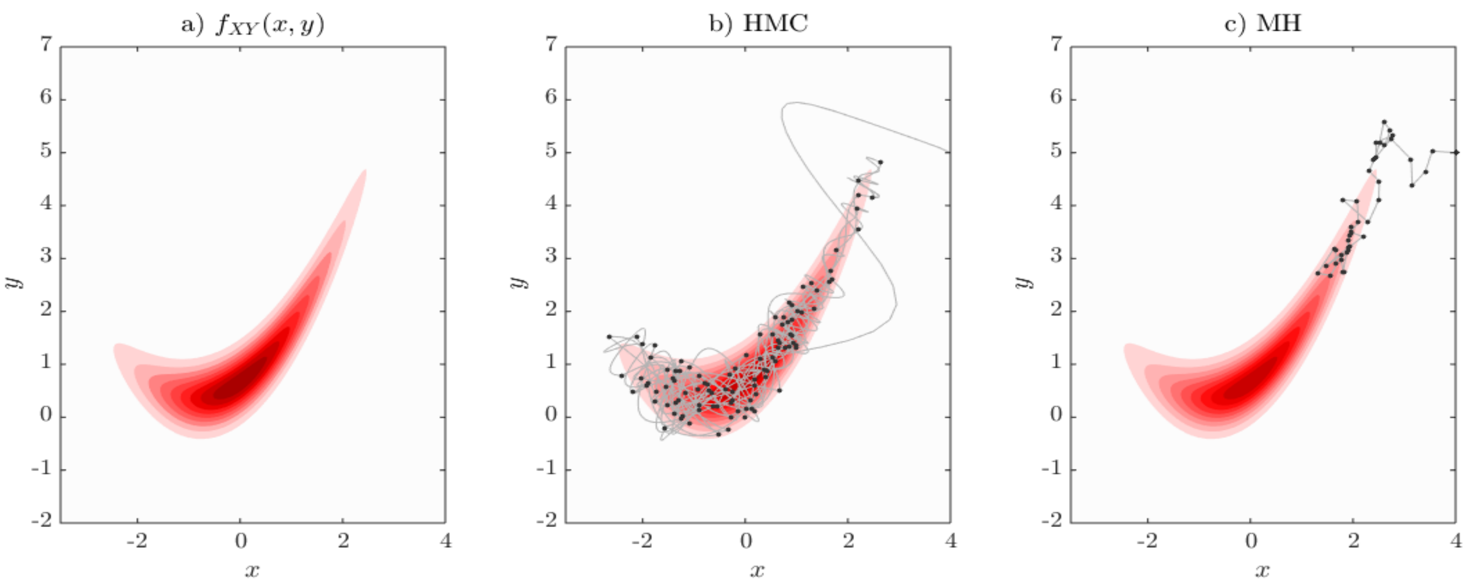

Important Ring
Data
ZO-HEM
FO-HEM


News·Today
WHAT'S
NEXT?

Future research

- **[Possibilities]** Integration with Hamiltonian Monte Carlo based subset simulation

Wang Z, Broccardo M, Song J. Hamiltonian Monte Carlo Methods for Subset Simulation in Reliability Analysis. arXiv:1706.01435



Summary

- **[Summary 1]** A hyper-spherical formulation to perform reliability analysis in high dimensional Gaussian space is proposed.
- **[Summary 2]** An importance sampling method using the hyper-spherical formulation in conjunction with von Mises-Fisher mixture distribution is proposed.
- **[Summary 3]** An extrapolation method using the the hyper-spherical formulation is proposed.

Wang, Z., and Song J.(2016). Cross-entropy-based adaptive importance sampling using von Mises–Fisher mixture for high dimensional reliability analysis. *Structural Safety*. 59: 42-52.

Wang, Z., and Song J. (2018). Hyper-spherical extrapolation method (HEM) for general high dimensional reliability problems. Structural Safety, 72: 65–73.

http://systemreliability.wordpress.com
junhosong@snu.ac.kr

Seoul National University
Department of Civil&Environmental Engineering

SSRG
Structural & System Reliability Group

Convergence Research Center
for Disaster-Hazard Resilience